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THE CAMBRIDGE AND DUBLIN

MATHEMATICAL JOURNAL.

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AND PROFESSOR OF NATURAL PHILOSOPHY IN THE UNIVERSITY OF GLASGOW.

VOL. I.

(BEING VOL. V. OF THE CAMBRIDGE MATHEMATICAL JOURNAL.)

Δυῶν ὀνομάτων μορφὴ μία.

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1846.

ERRATA.

The second member of equation (k), p. 206, should be 0.
The foot-note in p. 154, should be on (c).

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MECHANICS.

ASTRONOMY.

OPTICS.

HEAT, ELECTRICITY, AND MAGNETISM.

MISCELLANEOUS.

Notes and other passages enclosed in brackets [ ], are insertions made by the Editor.

The date of actual publication of any article, or portion of an article, may generally be found by reference to the first page of the sheet in which it is contained.—Ed.

we shall in effect integrate over the surface of the quadrilateral $OMAN'$. But if the two lines did not intersect within the positive quadrant, then one or other bounding inequality would be inoperative, and we should in effect integrate over the surface, not of a quadrilateral, but of a triangle, as in the case contemplated by Liouville's theorem. It is manifest that we may have, instead of two limiting inequalities, any larger number we please, and that our integrations may thus be made to extend over an irregular polygon of a greater or less number of sides. I do not believe that any writer on multiple integrals has considered the case in which the limits are given by more than one inequality, but the restriction to that of one is clearly unnecessary.

Let us suppose there are r variables $x, y \ldots . z$, and that we have to evaluate the integral

$$\int_0 dx \ldots \int_0 dz\, e^{-ax-\cdots-cz}\, \phi\,(mx + \ldots\, pz)\, \phi_{,}\,(m_{,}x + \ldots\, p_{,}z) \ldots . \;(1),$$

subject to the two inequalities

$$mx + \ldots . pz \leq h, \qquad m_{,}x + \ldots . p_{,}z \leq h_{,},$$

$m \ldots . p, h\,;\ m_{,} \ldots . p_{,}, h_{,}$ being all positive; and ϕ and $\phi_{,}$ any functions whose values may be represented within the limits of integration by Fourier's theorem.

Let the value of the integral in question be I; then, by considerations analogous to those of which I made use in a paper which appeared at the commencement of the last volume of the *Journal*, we shall have

$$I = \frac{1}{\pi^2}\int^{h} \phi u\, du \int^{h_{,}} \phi_{,}u_{,}\, du_{,} \int_0^\infty da \int_0^\infty da_{,} . G,$$

where

$$G = \int_0^\infty dx \ldots \int_0^\infty dz\, e^{-ax-\cdots-cz} \cos a(mx + \ldots pz - u) \cos a_{,}(m_{,}x + \ldots p_{,}z - u_{,}),$$

and the lower limits of integration with respect to u and u' may be any negative quantities.

I remark, in the first place, that

$$\int_0^\infty da \int_0^\infty da_1\; G = \tfrac{1}{4}\int_{-\infty}^\infty da \int_{-\infty}^\infty da_{,} H, \quad \text{where}$$

$$H = \int_0^\infty dx \ldots \int_0^\infty dz\, e^{-ax-\cdots-cz} \cos\{(am + a_{,}m_{,})x + \ldots (ap + a_{,}p_{,})z - au - a_{,}u_{,}\},$$

and therefore

$$I = \frac{1}{4\pi^2}\int^{h} \phi u\, du \int^{h_{,}} \phi_{,}u_{,}\, du_{,} \int_{-\infty}^\infty da \int_{-\infty}^\infty da_{,} H.$$

Let $\quad H = K \cos\,(au + a_{,}u_{,}) + L \sin\,(au + a_{,}u_{,}).$

Then it will be easily seen that

$$K = \frac{N}{D}, \quad L = \frac{N'}{D};$$

where, if we take the case of three variables,

$$N = abc\left(1 - \frac{am + a_{,}m_{,}}{a}\,\frac{an + a_{,}n_{,}}{b} - \frac{am + a_{,}m_{,}}{a}\,\frac{ap + a_{,}p_{,}}{c} - \frac{an + a_{,}n_{,}}{b}\,\frac{ap + a_{,}p_{,}}{c}\right),$$

$$N' = abc\left(\frac{am + a_{,}m_{,}}{a} + \frac{an + a_{,}n_{,}}{b} + \frac{ap + a_{,}p_{,}}{c} - \frac{am + a_{,}m_{,}}{a}\,\frac{an + a_{,}n_{,}}{b}\,\frac{ap + a_{,}p_{,}}{c}\right),$$

$$D = \{a^2 + (am + a_{,}m_{,})^2\}\,\{b^2 + (an + a_{,}n_{,})^2\}\,\{c^2 + (ap + a_{,}p_{,})^2\}.$$

(Precisely the same law of formation of these quantities would obtain if we were to take any number of variables. I have taken the case of three merely for distinctness of representation.)

Putting for $\cos(au + a_{,}u_{,})$ and $\sin(au + a_{,}u_{,})$ their exponential values, we find that

$$HD = a\ldots c\left\{1 - \sqrt{(-1)}\,\frac{am + a_{,}m_{,}}{a}\right\}\ldots\left\{1 - \sqrt{(-1)}\,\frac{ap + a_{,}p_{,}}{c}\right\}e^{(au + a_{,}u_{,})\sqrt{(-1)}}$$
$$+\, a\ldots c\left\{1 + \sqrt{(-1)}\,\frac{am + a_{,}m_{,}}{a}\right\}\ldots\left\{1 + \sqrt{(-1)}\,\frac{ap + a_{,}p_{,}}{c}\right\}e^{-(au + a_{,}u_{,})\sqrt{(-1)}};$$

and as

$$a^2 + (am + a_{,}m_{,})^2 = a^2\left\{1 - \sqrt{(-1)}\,\frac{am + a_{,}m_{,}}{a}\right\}\left\{1 + \sqrt{(-1)}\,\frac{am + a_{,}m_{,}}{a}\right\},$$

$$H = \frac{e^{(au + a_{,}u_{,})\sqrt{(-1)}}}{\{a + \sqrt{(-1)}\,(am + a_{,}m_{,})\}\ldots\{c + \sqrt{(-1)}\,(ap + a_{,}p_{,})\}} + \frac{e^{-(au + a_{,}u_{,})\sqrt{(-1)}}}{\{a - \sqrt{(-1)}\,(am + a_{,}m_{,})\}\ldots\{c - \sqrt{(-1)}\,(ap + a_{,}p_{,})\}}.$$

Now, assume that

$$\frac{1}{(a + am + a_{,}m_{,})\ldots(c + ap + a_{,}p_{,})}$$
$$= \frac{F_{ab}}{(a + am + a_{,}m_{,})(b + an + a_{,}n_{,})} + \frac{F_{ac}}{(a + am + a_{,}m_{,})(c + ap + a_{,}p_{,})} + \&c. \qquad \ldots\ldots (2);$$

where F_{ab}, F_{ac}, &c. are independent of a and $a_{,}$. This assumption is justifiable because it introduces $\frac{r.r-1}{2}$ disposable quantities F, viz. as many as there are combinations two and

two of the r quantities $a, b \ldots c$, and it will be easily seen that there are the same number of conditions to be satisfied.

Consequently as

$$e^{(au+a'u')\sqrt{(-1)}} = \cos(au + a'u') + \sqrt{(-1)}\sin(au + a'u'),$$

we shall have

$$\left.\begin{array}{l} H = F_{ab}\,\{ab - (am + a_{,}m_{,})(an + a_{,}n_{,})\}\cos(au + a_{,}u_{,}) \\ \qquad + \{a(an + a_{,}n_{,}) + b(am + a_{,}m_{,})\}\sin(an + a_{,}n_{,}) \\ \text{divided by} \\ \qquad \{a^2 + (am + a'm')^2\}\,\{b^2 + (an + a'n')^2\} \end{array}\right\} + \&c.$$

Let us next assume $u = mx + ny$, $u_{,} = m_{,}x + n_{,}y$, x and y being here two new variables; also $a' = am + a_{,}m_{,}$, and $\beta' = an + a_{,}n_{,}$: then the coefficient of F_{ab} in the expression of H will become

$$\frac{(ab - a'\beta')\cos(a'x + \beta'y) + (a\beta' + ba')\sin(a'x + \beta'y)}{(a^2 + a'^2)(b^2 + \beta'^2)}.$$

Moreover $du\,du_{1}\,da\,da_{1}$ will be replaced by $dx\,dy\,da'\,d\beta'$; and therefore, as we have

$$I = \frac{1}{4\pi^2}\int^h \phi u\,du \int^h \phi_{,}u_{,}\,du_{,} \int_{-\infty}^{+\infty} da \int_{-\infty}^{+\infty} da_{,}\,H, \text{ we shall have}$$

$$I = \frac{1}{4\pi^2}\,\Sigma F_{ab} \iint \phi\,(mx + ny)\,\phi_{,}(m_{,}x + n_{,}y)\,dx\,dy\,M,$$

where the sign of summation extends to all the quantities F, and where

$$M = \int_{-\infty}^{+\infty} da' \int_{-\infty}^{+\infty} d\beta'\,\frac{(ab - a'\beta')\cos(a'x + \beta'y) + (a\beta' + ba')\sin(a'x + \beta'y)}{(a^2 + a_{,}^2)(b^2 + \beta_{,}^2)}.$$

From the known integrals

$$\int_{-\infty}^{+\infty} \frac{\cos ax\,.\,da}{a^2 + a^2} = \frac{\pi}{a}\,e^{\mp ax} \quad \int_{-\infty}^{+\infty} \frac{a\sin ax\,.\,da}{a^2 + a^2} = \pm\,\pi e^{\mp ax},$$

the upper signs to be taken when x is positive, it follows that

$$M = \pi^2 e^{\mp ax \mp by}\,(1 \pm 1 \pm 1 \pm 1).$$

If x and y are both positive, the bracket becomes $1 + 1 + 1 + 1$ or 4; if x only be negative, it becomes $1 - 1 - 1 + 1$ or 0; if y only be negative, it becomes $1 - 1 + 1 - 1$ or 0; and similarly if both x and y are negative. Thus generally

$$M = 4\pi^2 e^{-ax-by} \text{ or } M = 0.$$

There are, indeed, exceptional cases; as if y be zero, x being positive, when $M = 2\pi^2 e^{-ax}$, and similarly if x be zero, y being

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Now a little consideration will convince us that

$$\int_0^\infty da_1 \dots \int_0^\infty da_s G = \frac{1}{2^s}\int_{-\infty}^{+\infty} da_1 \dots \int_{-\infty}^{+\infty} da_s H, \text{ where}$$

$$H = \int_0^\infty dx \dots \int_0^\infty dz\, e^{-ax-\dots-cz} \cos\left[x\Sigma am + \dots + z\Sigma ap - \Sigma au\right]:$$

for if we take the expression

$$\cos\left[x\Sigma am + \dots + z\Sigma ap - \Sigma au\right],$$

make a_1 negative, add the resulting expression to the original one: then in the two terms thus got make a_2 negative, and as before add the results, we shall, continuing this process, get in all 2^s terms, which will be found to be equal to 2^s times the continued product of the cosines involved in G.

Effecting the integrations indicated in H, we see that

$$H = \frac{N\cos\Sigma au + N'\sin\Sigma au}{D},$$

where $\qquad D = \{a^2 + (\Sigma am)^2\} \dots \{c^2 + (\Sigma ap)^2\},$

and N and N' follow the same law of formation as in the particular case already considered, except that for $\dfrac{am + a'm'}{a}$, &c. we substitute $\dfrac{\Sigma am}{a}$, &c. With this remark we perceive that

$$H = \frac{e^{\Sigma au\sqrt{(-1)}}}{\{a + \sqrt{(-1)}\,\Sigma am\}\dots\{c + \sqrt{(-1)}\,\Sigma ap\}}$$
$$+ \frac{e^{-\Sigma au\sqrt{(-1)}}}{\{a - \sqrt{(-1)}\,\Sigma am\}\dots\{c - \sqrt{(-1)}\,\Sigma ap\}} \dots\dots(3).$$

The assumption now to be made is that

$$\frac{1}{(a + \Sigma am)\dots(c + \Sigma ap)} = \Sigma\frac{F}{\Delta}\dots\dots(2'),$$

where Δ is the product of every set of s factors taken out of the whole number of r factors

$$a + \Sigma am, \quad \dots\ c + \Sigma ap,$$

and F is independent of $a_1 \dots a_s$.

There will thus be $\dfrac{r\,.\,r-1\dots r-s+1}{1\,.\,2\dots s}$ disposable quantities F, which will be found to be the number required to make (2′) identically true. Consequently we shall have

$$H = \Sigma F\frac{\nu\cos\Sigma au + \nu'\sin\Sigma au}{\delta},$$

any others, are so too. For if $x = -x'$, let its coefficient β_1 be assumed equal to $-\beta_1'$, when the expression of M' becomes of the same form as if x were positive, except that ν and ν' are changed by having $-\beta_1'$ wherever β_1 occurred previously. Now none of the quantities β can occur raised to any power, and therefore every term involving β_1 will change sign when β_1 is replaced by $-\beta_1'$. Hence we shall have

$$M_{,} = \pi^s e^{-ax'-by-\dots}(1 \pm 1 \dots .),$$

there being as many negative units as positive within the brackets, since in the development of $\sin(f_1 + \dots . f_s)$ or $\cos(f_1 + \dots . f_s)$ there are 2^{s-2} terms independent of the sine of f_1 and 2^{s-2} terms which involve that quantity, and which therefore change sign when f_1 does so. Hence the quantity within the bracket, and consequently $M_{,}$, is equal to zero if x be negative; and so, of course, for the other variables $y \dots z$.

M will, in particular cases analogous to those already noticed, assume exceptional or limiting values, but of these we need not take account. And thus we arrive at the following remarkable theorem:

The definite integral of r *variables* $\mathrm{x} \dots . \mathrm{z}$

$$\int_0 d\mathrm{x} \dots . \int_0 d\mathrm{z}\, \phi_1(\mathrm{m}_1\mathrm{x} + \dots . \mathrm{p}_1\mathrm{z}) \dots . \phi_s(\mathrm{m}_s\mathrm{x} + \dots . \mathrm{p}_s\mathrm{z})\, e^{-\mathrm{ax}-\dots \mathrm{cz}},$$

whose limits are given by s *inequalities*

$$\mathrm{m}_1\mathrm{x} + \dots . \mathrm{p}_1\mathrm{z} \leqq \mathrm{h}_1, \dots . \mathrm{m}_s\mathrm{x} + \dots . \mathrm{p}_s\mathrm{z} \leqq \mathrm{h}_s,$$

can generally be expressed as a linear function of

$$\frac{\mathrm{r}.(\mathrm{r}-1)\dots(\mathrm{r}-\mathrm{s}+1)}{1.2\dots \mathrm{s}}$$

integrals of s *variables each. The form of each of these integrals may be deduced from the original integral by omitting from it any set of* r − s *of the variables, and similarly the form of the limiting inequalities may be got by omitting the same set of variables from the original inequalities* (r > s).

In certain cases, however, when the constants a, m, &c. have particular values, the theorem fails because the assumption (2′) becomes illegitimate. This failure is indicated by certain of the quantities F becoming infinite. To determine the form of F, we have merely to multiply (2′) by Δ, and then to equate to zero all the s factors of which Δ is composed. All the quantities F, except the particular one under consideration, will then disappear, and we have s equations determining the s quantities a. Hence it will appear that

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as it is manifest that F will become $\dfrac{F}{k^{r-s}}$. Hence, if $\psi(ax+\ldots cz)$ be such a function that its development may be substituted for it in the integrations, we shall have

$$\int_0 dx \ldots \int_0 dz\, \phi_1(m_1x+\ldots p_1z)\ldots\phi_s(m_sx+\ldots p_sz)\,\psi(ax+\ldots cz)$$
$$= \Sigma F \int_0 dx \int_0 dy \ldots \phi_1(m_1x+n_1y+\ldots)\ldots\phi_s(m_sx+n_sy+\ldots)$$
$$\psi_1(ax+by+\ldots),$$

where $\dfrac{d^{r-s}}{dt^{r-s}}\psi_1 t = \psi t$ and all the differential coefficients of $\psi_1 t$ of an order lower than the $(r-s)^{\text{th}}$ vanish for $t=0$. This is, I believe, in the case of s equal to unity, precisely equivalent to one of Mr. Boole's results. It might also, I imagine, be obtained without having recourse to developments.

ON THE EQUATION OF LAPLACE'S FUNCTIONS.

By George Boole.

The partial differential equation of the second order, known as the Equation of Laplace's Functions, and usually expressed in the form

$$\frac{d}{d\mu}(1-\mu^2)\frac{du}{d\mu}+\frac{1}{1-\mu^2}\frac{d^2u}{d\phi^2}+n(n+1)u=0 \ldots\ldots(1),$$

is not more remarkable for the importance of its physical applications, than for the difficulties which it presents in a purely mathematical point of view. Mr. Hargreave, in the *Philosophical Transactions* for 1841, first succeeded in obtaining an expression for the complete integral. His analysis is original and most ingenious. Assuming two new variables, x and y, connected with the former ones by the relations

$$x=\phi+\tfrac{1}{2}\sqrt{-1}\log\frac{1+\mu}{1-\mu},\quad y=\phi-\tfrac{1}{2}\sqrt{-1}\log\frac{1+\mu}{1-\mu}\ldots(2),$$

he reduces the equation to the form

$$\frac{d^2u}{dx\,dy}+\frac{n(n+1)u}{4\cos^2\dfrac{x-y}{2}}=0 \ldots\ldots\ldots\ldots(3),$$

and, by a process of reduction which it is not necessary here to explain, he ultimately finds

[redacted]

As an example, let us take the equation

$$x^2\frac{d^2u}{dx^2}+q^2x^2u-6u=0 \quad \text{........ (8)},$$

which occurs in the theory of the earth's figure.

Let $x=\epsilon^\theta$. Now (Gregory's *Examples*, pp. 31, 32,)

$$x^2\frac{d^2}{dx^2}=D(D-1).$$

Hence
$$D(D-1)u+q^2\epsilon^{2\theta}u-6u=0,$$
$$\therefore\ (D+2)(D-3)u+q^2\epsilon^{2\theta}u=0,$$
$$u+\frac{q^2}{(D+2)(D-3)}\epsilon^{2\theta}u=0,$$

which is the symbolical form required.

From the symbolical form (7) we may at once deduce a theory of the solution of differential equations, in series extending to those cases in which the ordinary methods fail; but we shall here confine ourselves to two theorems, on which the *finite* solution of such equations chiefly depends.

Prop. 2. *When the equation* (7) *is of the form*

$$u+af(D)\epsilon^\theta u+bf(D)f(D-1)\epsilon^{2\theta}u+\&c.=0 \ \text{... (9)},$$

it may be resolved into a system of equations of the form

$$\left.\begin{aligned}u-q_1f(D)\epsilon^\theta u&=0,\\ u-q_2f(D)\epsilon^\theta u&=0,\end{aligned}\right\}\text{........(10)},$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot$$

q_1, q_2, *&c. being the roots of the equation*

$$q^n+aq^{n-1}+bq^{n-2}+\&c.=0 \ \text{...... (11)}.$$

To prove this, we observe, that if $f(D)\epsilon^\theta u=\rho u$, then

$$f(D)f(D-1)\epsilon^{2\theta}u=f(D)\epsilon^\theta f(D)\epsilon^\theta u=\rho^2u, \text{ by (6)},$$

and so on; whence (9) gives

$$(1+a\rho+b\rho^2+\&c.)u=0,$$

or
$$(1-q_1\rho)(1-q_2\rho)...u=0,$$

and the theorem is manifest. The reader will easily extend it to the case in which the equation has a second member. (*Phil. Trans.* 1844, p. 245.)

This case corresponds, in the present theory, to the case of

[redacted]

an equation of differences of the first order relative to $f(D)$, of which the solution is

$$f(D) = P_r \frac{\phi(D)}{\psi(D)};$$

whence $$u = P_r \frac{\phi(D)}{\psi(D)} v, \qquad U = P_r \frac{\phi(D)}{\psi(D)} V.$$

We proceed to exemplify our theory in the solution of the equation of Laplace's Functions.

We have

$$\frac{d}{d\mu}(1-\mu^2)\frac{du}{d\mu} + \frac{1}{1-\mu^2}\frac{d^2u}{d\phi^2} + n(n+1)\,u = 0 \,..(13).$$

Now ϕ only enters into this equation through the symbol of differentiation $\frac{d}{d\phi}$, which is commutative with respect to μ and $\frac{d}{d\mu}$. Let $\frac{d}{d\phi}\sqrt{-1} = a$, then

$$\frac{d}{d\mu}(1-\mu^2)\frac{du}{d\mu} - \frac{a^2}{1-\mu^2}u + n(n+1)\,u = 0 \dots (14).$$

If we can integrate this equation regarding a as a constant, and afterwards in the most general manner interpret our result, when for a we write $\frac{d}{d\phi}\sqrt{-1}$, we shall evidently be in possession of the complete integral required.

Now if in (14) we write $\mu = \epsilon^\theta$, and pass to the symbolical form, we shall have an equation involving three terms in the first member, and our analysis does not in its existing state possess any *general* method of treating such equations. We must therefore endeavour, by transforming the original equation, to obviate this difficulty.

The expanded form of (14) is

$$(1-\mu^2)\frac{d^2u}{d\mu^2} - 2\mu\frac{du}{d\mu} - \frac{a^2}{1-\mu^2}u + n(n+1)\,u = 0 \dots (15).$$

Assume $u = (1-\mu^2)^r v$, then

$$\frac{du}{d\mu} = (1-\mu^2)^r \frac{dv}{d\mu} - 2r\mu(1-\mu^2)^{r-1} v.$$

$$\frac{d^2u}{d\mu^2} = (1-\mu^2)^r \frac{d^2v}{d\mu^2} - 2r(1-\mu^2)^{r-1}\left(2\mu\frac{dv}{d\mu} + v\right)$$
$$+ 4r(r-1)\,\mu^2(1-\mu^2)^{r-2} v.$$

Substituting and effecting some reductions, we have

$$(1-\mu^2)^{r+1}\frac{d^2v}{d\mu^2}-(4r+2)\,\mu\,(1-\mu^2)^r\frac{dv}{d\mu}$$
$$+\left\{n(n+1)-2r-4r^2\right)(1-\mu^2)^r\,v+(4r^2-a^2)(1-\mu^2)^{r-1}v=0.$$

Let $4r^2-a^2=0$, then $r=\pm\frac{a}{2}$. Either sign may be taken; we choose the negative one, and suppose $r=-\frac{a}{2}$. Our equation now becomes, on dividing both sides by $(1-\mu^2)^r$,

$$(1-\mu^2)\frac{d^2v}{d\mu^2}+2(a-1)\mu\frac{dv}{d\mu}+\left\{n(n+1)-a(a-1)\right\}v=0\ldots(16).$$

In order to reduce this equation to the symbolical form, multiply by μ^2, then

$$(1-\mu^2)\,\mu^2\frac{d^2v}{d\mu^2}+2(a-1)\,\mu^3\frac{dv}{d\mu}+\left\{n(n+1)-a(a-1)\right\}\mu^2v=0.$$

Let $\mu=\varepsilon^\theta$, then $\mu\dfrac{d}{d\mu}=\dfrac{d}{d\theta}=D$, $\mu^2\dfrac{d^2}{d\mu^2}=D(D-1)$, whence

$$(1-\varepsilon^{2\theta})\,D(D-1)\,v+2(a-1)\varepsilon^{2\theta}Dv$$
$$+\left\{n(n+1)-a(a-1)\right\}\varepsilon^{2\theta}v=0\ldots\ldots(17).$$

Now $\varepsilon^{2\theta}D(D-1)\,v=(D-2)(D-3)\,\varepsilon^{2\theta}v$ by (6),
and $\varepsilon^{2\theta}Dv=(D-2)\,\varepsilon^{2\theta}v$.

Substituting these forms in (17), we have

$$D(D-1)\,v-\left\{(D-2)(D-3)-2(a-1)(D-2)-n(n+1)\right.$$
$$\left.+a(a-1)\right\}\varepsilon^{2\theta}v=0.$$

Or $D(D-1)\,v-\left\{D^2-(2a+3)D+a^2+3a-n^2-n+2\right\}\varepsilon^{2\theta}v=0.$

Or $\quad D(D-1)\,v-(D-a+n-1)(D-a-n-2)\,\varepsilon^{2\theta}v=0,$

on resolving the coefficient of $\varepsilon^{2\theta}v$ into its factors. Hence

$$v-\frac{(D-a+n-1)(D-a-n-2)}{D(D-1)}\,\varepsilon^{2\theta}v=0\ldots(18)$$

is the symbolical form required, and it is seen that the first member involves only two terms.

Now, let us assume

$$w-\frac{(D-a-n-1)(D-a-n-2)}{D(D-1)}\,\varepsilon^{2\theta}w=W\ldots\ldots(19).$$

In these two equations v and w respectively stand for u and v of Prop. 3.

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Hence $$\phi(D) = \frac{(D-a+n-1)(D-a-n-2)}{D(D-1)},$$

$$\psi(D) = \frac{(D-a-n-1)(D-a-n-2)}{D(D-1)};$$

$$\therefore P_r \frac{\phi(D)}{\psi(D)} = P_2 \frac{D-a+n-1}{D-a-n-1}$$

$$= (D-a+n-1)(D-a+n-3)\ldots(D-a-n+1);$$

$$\therefore v = (D-a+n-1)(D-a+n-3)\ldots\ldots(D-a-n+1)\,w,$$

$$o = (D-a+n-1)(D-a+n-3)\ldots\ldots(D-a-n+1)\,W.$$

The complete value of W determined from the last equation would be

$$W = c_1\varepsilon^{(a-n+1)\theta} + c_2\varepsilon^{(a-n+3)\theta} + \&c.;$$

but inasmuch as the transformed equation (19) is of the same degree as the original one (18), and will therefore introduce the requisite number of arbitrary constants, it is not necessary to retain any in W, so that we have simply $W = 0$: and it is remarkable that, were we to retain all the constants which the complete value of W involves, the final value of v would not be at all affected: the unnecessary constants would disappear. We have thus to consider the system,

$$v = (D-a+n-1)(D-a+n-3)\ldots(D-a-n+1)\,w \ldots (20),$$

$$w - \frac{(D-a-n-1)(D-a-n-2)}{D(D-1)}\varepsilon^{2\theta}w = 0 \ldots\ldots\ldots\ldots (21).$$

The last equation may, by Prop. 2, be resolved into the system

$$\left.\begin{aligned} w + \frac{D-a-n-1}{D}\varepsilon^{\theta}w = 0 \\ w - \frac{D-a-n-1}{D}\varepsilon^{\theta}w = 0 \end{aligned}\right\} \ldots\ldots(22).$$

From the former of these equations we have

$$Dw + (D-a-n-1)\varepsilon^{\theta}w = 0;$$

or $$Dw + \varepsilon^{\theta}(D-a-n)\,w = 0.$$

$$\therefore \frac{dw}{d\theta} - \frac{(a+n)\varepsilon^{\theta}}{1+\varepsilon^{\theta}}w = 0,$$

$$w = (1+\varepsilon^{\theta})^{n+a}\psi(\phi),$$

$\psi(\phi)$ denoting an arbitrary function of ϕ. In like manner, from the second equation of the system (22), we have

$$w = (1-\varepsilon^{\theta})^{n+a}\chi(\phi).$$

Hence the complete value of w is

$$w = (1 + \varepsilon^{\theta})^{n+a}\,\psi(\phi) + (1 - \varepsilon^{\theta})^{n+a}\,\chi(\phi)$$

$$= (1 + \mu)^{n+a}\,\psi(\phi) + (1 - \mu)^{n+a}\,\chi(\phi) \;\ldots\ldots (23).$$

To simplify the expression for v we observe that, in general,

$$f(D)\,U = f(D)\,\varepsilon^{r\theta}\,\varepsilon^{-r\theta}\,U$$

$$= \varepsilon^{r\theta} f(D + r)\,\varepsilon^{-r\theta}\,U \text{ by } (6).$$

Now inverting the order of the factors in the right-hand member of (20),

$$v = (D - a - n + 1)\,(D - a - n + 3)\,.\,.\,(D - a + n - 1)\,w$$

$$= \varepsilon^{(a+n-1)\theta} D(D + 2)\,.\,.\,(D + 2n - 2)\,\varepsilon^{-(a+n-1)\theta} w.$$

But $D + 2 = \varepsilon^{-2\theta} D\varepsilon^{2\theta}$, $D + 4 = \varepsilon^{-4\theta} D\varepsilon^{4\theta}$, and so on. Substituting, we have

$$v = \varepsilon^{(a+n-1)\theta}\, D\varepsilon^{-2\theta}\, D\varepsilon^{-2\theta} \ldots D\varepsilon^{(2n-2)\theta}\,\varepsilon^{-(a+n-1)\theta}\,w.$$

Making $\varepsilon^{\theta} = \mu$, $D = \mu \dfrac{d}{d\mu}$, the above equation assumes the form

$$v = \mu^{n+a}\left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^{n} \mu^{n-a}\,w \;\ldots\ldots\ldots\ldots (24).$$

Now $u = (1 - \mu^2)^{-\frac{a}{2}}\,v$; hence, writing in full the values of v and w, we have

$$u = (1-\mu^2)^{-\frac{a}{2}}\,\mu^{n+a}\left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^{n} \mu^{n-a}\left\{(1+\mu)^{n+a}\,\psi(\phi) + (1-\mu)^{n+a}\,\chi(\phi)\right\}$$

$$\ldots\ldots\ldots (25).$$

It remains to interpret this remarkable expression.

As $\psi(\phi)$, $\chi(\phi)$, are perfectly arbitrary, it is obvious that we may, in place of them, write $\psi(\epsilon^{\phi\sqrt{-1}})$, $\chi(\epsilon^{\phi\sqrt{-1}})$. The interpretation of our formula does not require this transformation, but the result is thereby simplified. We are thus at liberty to express the integral in the following form:

$$u = \mu^{n}\left(\frac{\mu}{\sqrt{(1 - \mu^2)}}\right)^{a}\left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^{n}\left\{(\mu + \mu^2)^{n}\left(\frac{1 + \mu}{\mu}\right)^{a}\psi(\epsilon^{\phi\sqrt{+1}})\right.$$

$$\left. + (\mu - \mu^2)^{n}\left(\frac{1 - \mu}{\mu}\right)^{a}\chi(\epsilon^{\phi\sqrt{-1}})\right\} \;\ldots\ldots (26),$$

or by two equations thus,

$$u = \left\{\frac{\mu}{\sqrt{(1 - \mu^2)}}\right\}^{a} F(\mu,\, \epsilon^{\phi\sqrt{-1}}),$$

[redacted]

where $F(\mu, \epsilon^{\phi\sqrt{-1}}) = \mu^n \left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^n \left\{(\mu + \mu^2)^n \left(\frac{1+\mu}{\mu}\right)^{\frac{d}{d\phi}\sqrt{-1}} \psi(\epsilon^{\phi\sqrt{-1}})\right.$

$$\left. + (\mu - \mu^2)^n \left(\frac{1-\mu}{\mu}\right)^{\frac{d}{d\phi}\sqrt{-1}} \chi(\epsilon^{\phi\sqrt{-1}})\right\} \ldots\ldots (27).$$

Now t being any quantity independent of ϕ, we have

$$\begin{aligned} t^{\frac{d}{d\phi}\sqrt{-1}} f(\epsilon^{\phi\sqrt{-1}}) &= t^{\frac{d}{d\phi}\sqrt{-1}} f(\epsilon^{\phi\sqrt{-1}}) \\ &= \epsilon^{\sqrt{-1}\log t\,\frac{d}{d\phi}} f(\epsilon^{\phi\sqrt{-1}}) \\ &= f(\epsilon^{(\phi+\sqrt{-1}\log t)\sqrt{-1}}), \text{ by Taylor's theorem,} \\ &= f\left(\frac{\epsilon^{\phi\sqrt{-1}}}{t}\right). \end{aligned}$$

Hence $\qquad \left(\frac{1+\mu}{\mu}\right)^{\frac{d}{d\phi}\sqrt{-1}} \psi(\epsilon^{\phi\sqrt{-1}}) = \psi\left(\frac{\mu\,\epsilon^{\phi\sqrt{-1}}}{1+\mu}\right),$

and so on; whence finally

$$u = F\left(\mu,\ \frac{\sqrt{(1-\mu^2)}}{\mu}\,\epsilon^{\phi\sqrt{-1}}\right),$$

where $F(\mu, \epsilon^{\phi\sqrt{-1}})$

$$= \mu^n \left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^n \left\{(\mu+\mu^2)^n\,\psi\left(\frac{\mu\,\epsilon^{\phi\sqrt{-1}}}{1+\mu}\right) + (\mu-\mu^2)^n\,\chi\left(\frac{\mu\,\epsilon^{\phi\sqrt{-1}}}{1-\mu}\right)\right\} \ldots (28),$$

which is the complete integral required. The symbol $\left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^n$, of course, implies the performance of n successive operations, each of which is effected by dividing the subject by μ, and then taking the differential coefficient with respect to that variable.

Discussion of the Integral.

If we represent the two particular integrals in the above general solution by U and V respectively, then in U, supposing $\psi(\epsilon^{\phi\sqrt{-1}}) = \epsilon^{r\phi\sqrt{-1}}$, we shall have

$$\begin{aligned} U &= \left\{\frac{\sqrt{(1-\mu^2)}\,\epsilon^{\phi\sqrt{-1}}}{\mu}\right\}^r \mu^n \left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^n (\mu+\mu^2)^n \left(\frac{\mu}{1+\mu}\right)^r \\ &= (1-\mu^2)^{\frac{r}{2}}\,\mu^{n-r} \left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^n \mu^{n+r}(1+\mu)^{n-r}\,\epsilon^{r\phi\sqrt{-1}} \\ &= f(\mu)\,\epsilon^{r\phi\sqrt{-1}}, \text{ for abbreviation} \ldots\ldots\ldots\ldots (29). \end{aligned}$$

[redacted]

and, by an exactly similar process of reasoning and comparison, we shall find that if, in the second integral V, we assume

$$\chi(\epsilon^{\phi\sqrt{-1}}) = a'\epsilon^{r\phi\sqrt{-1}} + b'\epsilon^{-r\phi\sqrt{-1}},$$

then
$$V = a'f(-\mu)\,\epsilon^{r\phi\sqrt{-1}} + (-)^r b'f(\mu)\,\epsilon^{-r\phi\sqrt{-1}} \ldots\ldots (36).$$

Adding these values,

$$\begin{aligned} u &= \{af(\mu) + a'f(-\mu)\}\,\epsilon^{r\phi\sqrt{-1}} + (-)^n\{b'f(\mu) + bf(-\mu)\}\,\epsilon^{-r\phi\sqrt{-1}} \\ &= \left.\begin{aligned} &[\{a + (-)^n b'\}f(\mu) + \{a' + (-)^r b\}f(-\mu)]\cos r\phi \\ &+ [\{a - (-)^n b'\}f(\mu) + \{a' - (-)^r b\}f(-\mu)]\sqrt{-1}\sin r\phi \end{aligned}\right\} \ldots (37). \end{aligned}$$

If we assume

$$a - (-)^r b' = 0, \qquad a' - (-)^r b = 0,$$

we have
$$u = 2\{af(\mu) + a'f(-\mu)\}\cos r\phi \ldots\ldots (38):$$

and as this is true for all values of r, it is clear that, had we assumed $\psi(\epsilon^{\phi\sqrt{-1}}) = \Sigma(a\epsilon^{r\phi\sqrt{-1}} + b\epsilon^{-r\phi\sqrt{-1}})$, &c., we should have had, in place of (38),

$$u = \Sigma\{2af(\mu) + 2a'f(-\mu)\}\cos r\phi;$$

or, putting $2a = c$, $2a' = c'$,

$$u = \Sigma\{cf(\mu) + c'f(-\mu)\}\cos r\phi \ldots\ldots (39),$$

the values of c and c' differing for different values of r.

This is the most general form, *of the kind*, which the integral can assume. It is remarkable that the coefficient of $\cos r\phi$ will be always a *finite algebraic* function of μ, whether r be integral or not, its expression being

$$f(\mu) = (1 - \mu^2)^{\frac{r}{2}}\,\mu^{n-r}\left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^n \mu^{n+r}\,(1 + \mu)^{n-r} \ldots\ldots (40).$$

It is also to be remarked that, by attributing other forms to ψ and χ, and especially *logarithmic* forms, we can obtain an infinite variety of finite solutions of a character altogether different from the above.

Forms of Laplace's Coefficients.

In the case we have now to consider, u, or as it is commonly written, P_n, is the coefficient of t^n in the development of the function

$$[1 - 2\{\mu\mu' + \sqrt{(1 - \mu^2)(1 - \mu'^2)}\cos\phi\}\,t + t^2]^{-\frac{1}{2}} \ldots (41),$$

ϕ standing for $\phi - \phi'$ in the ordinary treatises.

[redacted]

If $n - r$ be odd, the first term of the expansion of $a_r f(\mu) f(\mu')$ will be $a_r \{(r+n)(r+n-2)\ldots(r-n)\}^2$; which, equated with the corresponding term in (42), leads to the same result.

Hence, finally, we have

$$P_n = A_0 + 2(A_1 \cos\phi + A_2 \cos 2\phi \ldots + A_n \cos n\phi) \ldots (44),$$

where any coefficient A_r is of the form

$$A_r = \frac{f(\mu) f(\mu')}{1.2 \ldots n+r \;\; 1.2 \ldots n-r},$$

and in general

$$f(\mu) = (1 - \mu^2)^{\frac{r}{2}} \mu^{n-r} \left(\frac{d}{d\mu} \frac{1}{\mu}\right)^n \mu^{n+r} (1 + \mu)^{n-r}.$$

I have entered with more particularity into the details of the above solution, than to some might have appeared necessary; but it was my object in this paper, not only to integrate the Equation of Laplace, but also to illustrate, and in so doing, if it might be, to recommend a method in Analysis.

Lincoln, July 28, 1845.

ON SOME ANALYTICAL FORMULÆ, AND THEIR APPLICATION TO THE THEORY OF SPHERICAL CO-ORDINATES.

By ARTHUR CAYLEY, M.A., Fellow of Trinity College, Cambridge.

Section 1.

THE formulæ in question are only very particular cases of some relating to the theory of the transformation of functions of the second order, which will be given in a following paper. But the case of three variables, here as elsewhere, admits of a symmetrical notation so much simpler than in any other case (on the principle that with three quantities a, b, c, functions of b, c; of c, a; and a, b, may symmetrically be denoted by A, B, C, which is not possible with a greater number of variables) that it will be convenient to employ here a notation entirely different from that made use of in the general case, and by means of which the results will be exhibited in a more compact form. There is no difficulty in verifying by actual multiplication, any of the equations here obtained.

It will be expedient to employ the abbreviation of making a single letter stand for a system of quantities. Thus for instance, if $\aleph = \theta, \phi, \psi$, this merely means that $\Phi(\aleph)$ is to stand for $\Phi(\theta, \phi, \psi)$, $k\aleph$ for $k\theta, k\phi, k\psi$, &c.

[redacted]

Suppose then
$$\omega = \xi, \eta, \zeta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$
$$\omega' = \xi', \eta', \zeta',$$
$$\cdot$$
$$Q = A, B, C, F, G, H. \quad . \quad . \quad . \quad (2),$$
$$W(\omega, \omega', Q) = A\xi\xi' + B\eta\eta' + C\zeta\zeta'$$
$$+ F(\eta\zeta' + \eta'\zeta) + G(\zeta\xi' + \zeta'\xi) + H(\xi\eta' + \xi'\eta) \ldots (3).$$

The function W satisfies a remarkable equation, as follows.

Write
$$\mathfrak{A} = BC - F^2, \quad . \quad . \quad . \quad . \quad . \quad (4),$$
$$\mathfrak{B} = CA - G^2,$$
$$\mathfrak{C} = AB - H^2.$$
$$\mathfrak{F} = GH - AF,$$
$$\mathfrak{G} = HF - BG,$$
$$\mathfrak{H} = FG - CH.$$
$$\mathfrak{Q} = \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \quad . \quad . \quad . \quad . \quad (5).$$
$$\overline{\omega\omega'} = \eta\zeta' - \eta'\zeta, \ \zeta\xi' - \zeta'\xi, \ \xi\eta' - \xi'\eta \quad . \quad . \quad (6).$$

We have
$$W(\omega_1, \omega_2, Q)\ W(\omega_3, \omega_4, Q) - W(\omega_1, \omega_3, Q)\ W(\omega_2, \omega_4, Q)$$
$$= W(\overline{\omega_1\omega_4}, \overline{\omega_2\omega_3}, \mathfrak{Q}) \ldots (7),$$

of which we may notice also the particular cases
$$W(\omega_1\omega_2, Q)\ W(\omega_3\omega_3, Q) - W(\omega_1\omega_3, Q)\ W(\omega_2, \omega_3, Q)$$
$$= W(\overline{\omega_1\omega_3}, \overline{\omega_2\omega_3}, \mathfrak{Q}) \, . \, . (8),$$
$$W(\omega_1, \omega_1, Q)\ W(\omega_2, \omega_2, Q) - \{W(\omega_1\omega_2, Q)\}^2$$
$$= W(\overline{\omega_1\omega_2}, \overline{\omega_1\omega_2}, \mathfrak{Q}) \, . \, . (9).$$

To which we may join the following formulæ, for the transformation of the function W.

Suppose
$$\omega_1 = \mathrm{a}x_1 + \mathrm{a}'y_1 + \mathrm{a}''z_1, \ \mathrm{b}x_1 + \mathrm{b}'y_1 + \mathrm{b}''z_1, \ \mathrm{c}x_1 + \mathrm{c}'y_1 + \mathrm{c}''z_1 \, . \, . (10).$$
$$\omega_2 = \mathrm{a}x_2 + \mathrm{a}'y_2 + \mathrm{a}''z_2, \ \mathrm{b}x_2 + \mathrm{b}'y_2 + \mathrm{b}''z_2, \ \mathrm{c}x_2 + \mathrm{c}'y_2 + \mathrm{c}''z_2.$$
$$\cdot$$

Then, writing
$$g = \mathrm{a}, \mathrm{b}, \mathrm{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$
$$g' = \mathrm{a}', \mathrm{b}', \mathrm{c}',$$
$$g'' = \mathrm{a}'', \mathrm{b}'', \mathrm{c}''.$$
$$p_1 = x_1, y_1, z_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$
$$p_2 = x_2, y_2, z_2,$$
$$\cdot$$
$$\Theta = W(g, g, Q), \ W(g', g', Q), \ W(g'', g'', Q),$$
$$W(g', g'', Q), \ W(g'', g, Q), \ W(g, g', Q) \, . \, . (13).$$

We have $$W(\omega_1, \omega_2, Q) = W(p_1, p_2, \Theta) \quad . \quad . \quad . \quad (14).$$
Similarly, writing

$$\Psi = W(\overline{\underline{g'g''}}, \overline{\underline{g'g''}}, \Theta),\ W(\overline{\underline{g''g}}, \overline{\underline{g''g}}, \Theta),\ W(\overline{\underline{gg'}}, \overline{\underline{gg'}}, \Theta) \,..\, (15).$$

$$W(\overline{\underline{gg'}}, \overline{\underline{g'g}}, \Theta),\quad W(\overline{\underline{g'g''}}, \overline{\underline{gg'}}, \Theta),\quad W(\overline{\underline{g''g}}, \overline{\underline{g'g''}}, \Theta).$$

we have $$W(\overline{\underline{\omega_1\omega_4}}, \overline{\underline{\omega_2\omega_3}}, \Theta) = W(\overline{\underline{p_1p_4}}, \overline{\underline{p_2p_3}}, \Psi) \ldots (16),$$

in which equations Θ may obviously be changed into Q.

Section 2.—*Geometrical Applications.*

Consider any three axes Ax, Ay, Az, and let λ, μ, ν be the cosines of the inclinations of these lines to each other.

Let Λ, M, N be the inclinations of the co-ordinate planes to each other; l, m, n, the inclination of the axes to the co-ordinate planes. Suppose, besides,

$$\begin{aligned} \mathfrak{a} &= 1 - \lambda^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17). \\ \mathfrak{b} &= 1 - \mu^2, \\ \mathfrak{c} &= 1 - \nu^2, \\ \mathfrak{f} &= \mu\nu - \lambda, \\ \mathfrak{g} &= \nu\lambda - \mu, \\ \mathfrak{h} &= \lambda\mu - \nu. \end{aligned}$$

$$k = 1 - \lambda^2 - \mu^2 - \nu^2 + 2\lambda\mu\nu \quad . \quad . \quad . \quad (18).$$

We have the following systems of equations:

$$\begin{aligned} &\sqrt{(\mathfrak{bc})} \cos \Lambda = -\mathfrak{f}, && \sqrt{(\mathfrak{bc})} \sin \Lambda = \sqrt{(k)}, && \sqrt{(\mathfrak{a})} \sin l = \sqrt{(k)} \ldots (19). \\ &\sqrt{(\mathfrak{ca})} \cos \mathrm{M} = -\mathfrak{g}, && \sqrt{(\mathfrak{ca})} \sin \mathrm{M} = \sqrt{(k)}, && \sqrt{(\mathfrak{b})} \sin m = \sqrt{(k)} \\ &\sqrt{(\mathfrak{ab})} \cos \mathrm{N} = -\mathfrak{h}, && \sqrt{(\mathfrak{ab})} \sin \mathrm{N} = \sqrt{(k)}, && \sqrt{(\mathfrak{c})} \sin n = \sqrt{(k)}. \end{aligned}$$

$$\begin{aligned} \mathfrak{a} + \nu\mathfrak{h} + \mu\mathfrak{g} &= k, \quad . \quad . \quad . \quad . \quad . \quad (20). \\ \nu\mathfrak{a} + \mathfrak{h} + \lambda\mathfrak{g} &= 0, \\ \mu\mathfrak{a} + \lambda\mathfrak{h} + \mathfrak{g} &= 0, \end{aligned}$$

$$\begin{aligned} \mathfrak{h} + \nu\mathfrak{b} + \mu\mathfrak{f} &= 0, \quad . \quad . \quad . \quad . \quad . \quad (21). \\ \nu\mathfrak{h} + \mathfrak{b} + \lambda\mathfrak{f} &= k, \\ \mu\mathfrak{h} + \lambda\mathfrak{b} + \mathfrak{f} &= 0. \end{aligned}$$

$$\begin{aligned} \mathfrak{g} + \nu\mathfrak{f} + \mu\mathfrak{c} &= 0, \quad . \quad . \quad . \quad . \quad . \quad (22). \\ \nu\mathfrak{g} + \mathfrak{f} + \lambda\mathfrak{c} &= 0, \\ \mu\mathfrak{g} + \lambda\mathfrak{f} + \mathfrak{c} &= k. \end{aligned}$$

Let AO' be any other line, and δ its inclination to AO: $\alpha', \beta', \gamma', a', b', c'$, the quantities corresponding to $\alpha, \beta, \gamma, a, b, c$, and similarly t', τ' to t, τ. We have of course

$$1 = W(t', t', \mathfrak{q}) \quad . \quad . \quad . \quad . \quad (38),$$

$$k = W(\tau', \tau', \mathfrak{q}) \quad . \quad . \quad . \quad . \quad (39).$$

We have besides, by projecting on the line AO', the equation

$$\cos\delta = \alpha' a + \beta' b + \gamma' c \quad . \quad . \quad . \quad (40),$$

or the analogous one

$$\cos\delta = \alpha' a + \beta' b + \gamma' c \quad . \quad . \quad . \quad (41).$$

From either of which we deduce

$$\cos\delta = aa' + bb' + cc' + \lambda.(bc' + b'c) + \mu.(ca' + c'a) + \nu.(ab' + a'b) \ldots (42),$$

$$k\cos\delta = \mathfrak{a}\alpha\alpha' + \mathfrak{b}\beta\beta' + \mathfrak{c}\gamma\gamma' + \mathfrak{f}(\beta\gamma' + \beta'\gamma) + \mathfrak{g}(\gamma\alpha' + \gamma'\alpha) + \mathfrak{h}(\alpha\beta' + \alpha'\beta) \ldots (43);$$

which may otherwise be written

$$\cos\delta = W(t, t', \mathfrak{q}) \quad . \quad . \quad . \quad . \quad (44),$$

$$k\cos\delta = W(\tau, \tau', \mathfrak{q}) \quad . \quad . \quad . \quad . \quad (45).$$

Or again, observing the equations which connect the quantities t, τ,

$$\cos\delta = \frac{W(t, t', \mathfrak{q})}{\sqrt{\{W(t, t, \mathfrak{q}) . W(t', t', \mathfrak{q})\}}} \quad . \quad . \quad (46),$$

$$\cos\delta = \frac{W(\tau, \tau', \mathfrak{q})}{\sqrt{\{W(\tau, \tau, \mathfrak{q}) . W\tau', \tau', \mathfrak{q}\}}} \quad . \quad . \quad (47),$$

forms which, though more complicated, have certain advantages; for instance, we derive immediately from them the new equations

$$\sin\delta = \frac{\sqrt{\{W(\overline{tt'}, \overline{tt'}, \mathfrak{q})\}}}{\sqrt{\{W(t, t, \mathfrak{q})\, W(t', t', \mathfrak{q})\}}} \quad . \quad . \quad (48),$$

$$\sin\delta = \frac{\sqrt{\{k W(\overline{\tau\tau'}, \overline{\tau\tau'}, \mathfrak{q})\}}}{\sqrt{\{W(\tau, \tau, \mathfrak{q})\, W(\tau', \tau', \mathfrak{q})\}}} \quad . \quad . \quad (49).$$

Written more simply thus

$$\sin\delta = W(\overline{tt'}, \overline{tt'}, \mathfrak{q}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50),$$

$$\sqrt{k}\sin\delta = \sqrt{\{W(\overline{\tau\tau'}, \overline{\tau\tau'}, \mathfrak{q})\}} \quad . \quad . \quad . \quad . \quad (51),$$

to which we may join

$$\cot\delta = \frac{W(t, t', \mathfrak{q})}{\sqrt{\{W(\overline{tt'}, \overline{tt'}, \mathfrak{q})\}}} \quad . \quad . \quad . \quad . \quad . \quad (52),$$

[redacted]

Fundamental formula of spherical coordinates; distance of two points.

Let P, P' be the points, δ their distance, ω, p the conjoint coordinate systems of the first point, ω', p' of the second; we have obviously

$$\cos\delta = \frac{W(p, p', \mathfrak{q})}{\sqrt{\{W(p, p, \mathfrak{q})\, W(p', p', \mathfrak{q})\}}} \;\ldots\ldots\; (61),$$

$$\sin\delta = \frac{\sqrt{\{W(\overline{pp'}, \overline{pp'}, \mathfrak{q})\}}}{\sqrt{\{W(p, p, \mathfrak{q})\, W(p', p', \mathfrak{q})\}}},$$

$$\cot\delta = \frac{W(p, p', \mathfrak{q})}{\sqrt{\{W(\overline{pp'}, \overline{pp'}, \mathfrak{q})\}}};$$

or

$$\cos\delta = \frac{W(\omega, \omega', \mathfrak{q})}{\sqrt{\{W(\omega, \omega, \mathfrak{q})\, W(\omega', \omega', \mathfrak{q})\}}} \;\ldots\ldots\; (62).$$

$$\frac{1}{\sqrt{k}}\sin\delta = \frac{\sqrt{\{W(\overline{\omega\omega'}, \overline{\omega\omega'}, \mathfrak{q})\}}}{\sqrt{\{W(\omega, \omega, \mathfrak{q})\, W(\omega', \omega', \mathfrak{q})\}}},$$

$$\sqrt{k}\cot\delta = \frac{W(\omega, \omega')\, \mathfrak{q})}{\sqrt{\{W(\overline{\omega\omega'}, \overline{\omega\omega'}, \mathfrak{q})\}}}.$$

Equation of a great Circle.

Let the conjoint coordinate systems of the pole be

$$e = a, b, c \;\ldots\ldots\ldots\ldots\; (63),$$

$$\epsilon = \alpha, \beta, \gamma \;\ldots\ldots\ldots\ldots\; (64).$$

Then, expressing that the distance of any point P in the locus from the pole is equal to 90°, we have immediately the equations

$$W(p, e, \mathfrak{q}) = 0 \;\ldots\ldots\ldots\ldots\; (65),$$

$$W(\omega, \epsilon, \mathfrak{q}) = 0 \;\ldots\ldots\ldots\ldots\; (66),$$

which may otherwise be written in the forms

$$a\xi + b\eta + c\zeta = 0 \;\ldots\ldots\ldots\ldots\; (67),$$

$$\alpha x + \beta y + \gamma\zeta = 0 \;\ldots\ldots\ldots\ldots\; (68),$$

or the equation of a great circle is linear in either coordinate system. Conversely, any linear equation belongs to a great circle.

Suppose the equation given in the form

$$A\xi + B\eta + C\zeta = 0 \;\ldots\ldots\ldots\; (69);$$

or by an equation between cosine coordinate ratios:—The sine system for the pole is given by

$$e = A, B, C \;\ldots\ldots\ldots\ldots\; (70),$$

[redacted]

$$\mathfrak{e}_1 = \frac{W(\overline{ee'}, \overline{ee'}, \mathfrak{q})}{W(e, e, \mathrm{q})\,W(e', e', \mathrm{q})} = \frac{kW(\overline{\epsilon\epsilon'}, \overline{\epsilon\epsilon'}, \mathrm{q})}{W(\epsilon, \epsilon, \mathfrak{q})\,W(\epsilon', \epsilon', \mathfrak{q})},$$

$$\mathfrak{f}_1 = \frac{W(\overline{e''e}, \overline{ee'}, \mathfrak{q})}{W(e, e, \mathrm{q})\,\sqrt{\{W(e', e', \mathrm{q})\,W(e'', e'', \mathrm{q})\}}}$$
$$= \frac{kW(\overline{\epsilon''\epsilon}, \overline{\epsilon\epsilon'}, \mathrm{q})}{W(\epsilon, \epsilon, \mathfrak{q})\,\sqrt{\{W(\epsilon', \epsilon', \mathfrak{q})\,W(\epsilon'', \epsilon'', \mathfrak{q})\}}},$$

$$\mathfrak{g}_1 = \frac{W(\overline{ee'}, \overline{e'e''}, \mathfrak{q})}{W(e', e', \mathrm{q})\,\sqrt{\{W(e'', e'', \mathrm{q})\,W(e, e, \mathrm{q})\}}}$$
$$= \frac{kW(\overline{\epsilon\epsilon'}, \overline{\epsilon'\epsilon''}, \mathrm{q})}{W(\epsilon', \epsilon', \mathfrak{q})\,\sqrt{\{W(\epsilon'', \epsilon'', \mathfrak{q})\,W(\epsilon, \epsilon, \mathfrak{q})\}}},$$

$$\mathfrak{h}_1 = \frac{W(\overline{e'e''}, \overline{e''e}, \mathfrak{q})}{W(e'', e'', \mathrm{q})\,\sqrt{\{W(e, e, \mathrm{q})\,W(e', e', \mathrm{q})\}}}$$
$$= \frac{kW(\overline{\epsilon'\epsilon''}, \overline{\epsilon''\epsilon}, \mathrm{q})}{W(\epsilon'', \epsilon'', \mathfrak{q})\,\sqrt{\{W(\epsilon, \epsilon, \mathfrak{q})\,W(\epsilon', \epsilon', \mathfrak{q})\}}}.$$

$$\begin{aligned} x_1 : y_1 : z_1 = {} & \sqrt{\{W(e, e, \mathrm{q})\}}\,\{W(e, p, \mathrm{q})\,W(\overline{e'e''}, \overline{e'e''}, \mathfrak{q}) \\ & + W(e', p, \mathrm{q})\,W(\overline{e'e''}, \overline{e''e}, \mathfrak{q}) + W(e'', p, \mathfrak{q})\,W(\overline{e'e''}, \overline{ee'}, \mathfrak{q})\} \quad \ldots(90), \\ & : \sqrt{\{W(e', e', \mathrm{q})\}}\,\{W(e, p, \mathrm{q})\,W(\overline{e''e}, \overline{e'e''}, \mathfrak{q}) \\ & + W(e', p, \mathrm{q})\,W(\overline{e''e}, \overline{e''e}, \mathfrak{q}) + W(e'', p, \mathrm{q})\,W(\overline{e''e}, \overline{ee'}, \mathrm{q})\} \\ & : \sqrt{\{W(e'', e'', \mathrm{q})\}}\,\{W(e, p, \mathrm{q})\,W(\overline{ee'}, \overline{e'e''}, \mathfrak{q}) \\ & + W(e', p, \mathrm{q})\,W(\overline{ee'}, \overline{e''e}, \mathfrak{q}) + W(e'', p, \mathrm{q})\,W(\overline{ee'}, \overline{ee'}, \mathfrak{q})\}; \end{aligned}$$

which may be reduced to the very simple form

$$\begin{aligned} x_1 : y_1 : z_1 = {} & \sqrt{\{W(e, e, \mathrm{q})\}}\,W(\overline{e'e''}, \omega, \mathfrak{q}) \quad \ldots\ldots (91), \\ & : \sqrt{\{W(e', e', \mathrm{q})\}}\,W(\overline{e''e}, \omega, \mathfrak{q}), \\ & : \sqrt{\{W(e'', e'', \mathrm{q})\}}\,W(\overline{ee'}, \omega, \mathfrak{q}). \end{aligned}$$

And in like manner we obtain

$$\begin{aligned} x_1 : y_1 : z_1 = {} & \sqrt{\{W(\epsilon, \epsilon, \mathfrak{q})\}}\,W(\overline{\epsilon'\epsilon''}, p, \mathrm{q}) \quad \ldots\ldots (92), \\ & : \sqrt{\{W(\epsilon', \epsilon', \mathfrak{q})\}}\,W(\overline{\epsilon''\epsilon}, p, \mathrm{q}), \\ & : \sqrt{\{W(\epsilon'', \epsilon'', \mathfrak{q})\}}\,W(\overline{\epsilon\epsilon'}, p, \mathrm{q}). \end{aligned}$$

It will be as well to indicate the steps of this reduction. Consider the quantity in { } in the first line of the equation which gives the ratios $x_1 : y_1 : z_1$; and suppose for a moment $\overline{e'e''} = l, m, n$, &c., selecting the portion of the expression which is multiplied by $\mathfrak{a}$, this is

$$\mathfrak{a}l\,.\,\{l(a\xi + b\eta + c\zeta) + l'(a'\xi + b'\eta + c'\zeta) + l''(a''\xi + b''\eta + c''\zeta)\}.$$

Or, since

$$la + l'a' + l''a'' = \overline{ee'e''}, \quad lb + l'b' + l''b'' = 0, \quad lc + l'c' + l''c'' = 0,$$

this reduces itself to $\overline{ee'e''}$, $\mathfrak{a}l\xi$, which is a term of

$$\overline{ee'e''}\,W(\overline{e'e''}, \omega, \mathfrak{q});$$

and by comparing the remaining terms in the same manner, it would be seen that the whole reduces itself to

$$\overline{ee'e''}\,W(\overline{e'e''}, \omega, \mathfrak{q});$$

whence the formulæ in question.

The formulæ (86), (87), (91), (92), completely resolve the problem of the transformation of coordinates; they determine respectively p_1 from p or ω, ω_1 from p or ω.

To complete the present part of the subject we may add the following formulæ.

Suppose $$x_1 : y_1 : z_1 = \mathrm{a}x_1 + \mathrm{a}'y_1 + \mathrm{a}''z_1 \; \ldots\ldots\ldots\ldots \; (93),$$
$$: \mathrm{b}x_1 + \mathrm{b}'y_1 + \mathrm{b}''y_1,$$
$$: \mathrm{c}x_1 + \mathrm{c}'y_1 + \mathrm{c}''z_1,$$

which we see from the preceding formulæ is the form of the relation between the systems p_1 and p. And suppose, as before, λ_1, μ_1, ν_1 are the cosines of the distances of the new points of reference X_1, Y_1, Z_1.

We can immediately determine the relations that must exist between these coefficients, in order that they may actually denote such a transformation. For this purpose write

$$\mathrm{a}, \mathrm{b}, \mathrm{c} = j \; \ldots\ldots\ldots\ldots\ldots \; (94),$$
$$\mathrm{a}', \mathrm{b}', \mathrm{c}' = j',$$
$$\mathrm{a}'', \mathrm{b}'', \mathrm{c}'' = j''.$$

Then the distance between the point P and any other point P' is given by the formula

$$\cos\delta = \frac{W(p, p', \mathfrak{q})}{\sqrt{\{W(p, p, \mathfrak{q})\,W(p', p', \mathfrak{q})\}}}$$
$$= \frac{W(p_1, p_1', \Theta)}{\sqrt{\{W(p_1, p_1, \Theta)\,W(p_1, p_1', \Theta)\}}} \;\ldots\ldots\; (95),$$

[redacted]

In the case when the plate is of iron, the value of m is nearly unity. Hence, as the series is multiplied by $1-m^2$, it might be imagined that, if we "neglect small quantities of the order $(1-g)$ compared with those which are retained," ($1-g$ being, in Green's notation, a quantity of the same order as $1-m$), an approximate result would be obtained by putting $m=1$ in the successive terms of the series within the vinculum. And it is thus that Green, having, in the investigation, neglected quantities multiplied by $(1-g)^2$, arrives at the result,

$$F=\frac{4(1-g)}{3}\left\{\frac{1}{(x^2+y)^{2\frac{1}{2}}}+\frac{1}{\{(x+2a)^2+y^2\}^{\frac{1}{2}}}+\frac{1}{\{(x+4a)^2+y^2\}^{\frac{1}{2}}}+\&c.\right\}$$

As, however, this series has an infinite sum, it is clear that no value of m can be sufficiently near to unity to render the approximation admissible. If instead of Q we were to substitute a magnet, or any collection of positive and negative particles, such that the sum of the masses is zero, the series for the potential, deduced from Green's expression, would converge: and the same remark is applicable to the series which would be found for the *attraction* of the system on a point beyond the screen, even when Q is a positive point, by differentiating the expression for F. Notwithstanding this, the approximation is still inadmissible; since, if we expand the rigorous expression in either case in ascending powers $(1-m)$, we find that, though the first term is finite, the coefficients of all the terms which follow it are infinite.

Although the method by which I obtained the rigorous solution is quite distinct from that followed by Green, being independent of any mathematical process, it may be satisfactory to shew that the result can be deduced from his own analysis, and even with greater ease than his solution is obtained, after making unnecessary approximation.

By a very remarkable investigation, in which he extends Laplace's well-known analysis for spherical coordinates to the case when the radius of the sphere becomes infinite, Green arrives (*Essay on Electricity*, p. 64) at the following expression for the total potential at P, due to the positive unit of matter Q, and to the interposed plate, before making any approximation:

$$F=\frac{8}{\pi}(1-g)(1+2g)\int_0^{\infty}\frac{d\gamma\,\epsilon^{-\gamma x}}{(2+g)^2-9g^2\epsilon^{-2\gamma a}}\int_0^1\frac{d\beta}{(1-\beta^2)^{\frac{1}{2}}}\cos(\beta\gamma y).$$

[redacted]

$$l\frac{dx}{dt}+m\frac{dy}{dt}+n\frac{dz}{dt}=0\,\ldots\ldots\ldots\ldots(1),$$

$$x\frac{dl}{dt}+y\frac{dm}{dt}+z\frac{dn}{dt}=0\,\ldots\ldots\ldots\ldots(2),$$

$$l\frac{dl}{dt}+m\frac{dm}{dt}+n\frac{dn}{dt}=0\,\ldots\ldots\ldots\ldots(3).$$

And differentiating (1) again, we have by equations (I)

$$\frac{dl}{dt}\,\frac{dx}{dt}+\frac{dm}{dt}\,\frac{dy}{dt}+\frac{dx}{dt}\,\frac{dz}{dt}=-\left(l\frac{d^2x}{dt^2}+m\frac{d^2y}{dt^2}+n\frac{d^2z}{dt^2}\right)$$

$$=-\left(l\frac{dR}{dx}+m\frac{dR}{dy}+n\frac{dR}{dz}\right)=-\frac{dR}{d\zeta}=-S\ldots(4),$$

where S is the force perpendicular to the plane of the orbit.

Combining (2) and (3) with this last, we get

$$\frac{\frac{dl}{dt}}{mz-ny}=\frac{\frac{dm}{dt}}{nx-lz}=\frac{\frac{dn}{dt}}{ly-mx}=\frac{-\frac{dR}{d\zeta}}{H}=\frac{-S}{H}\ldots(5),$$

since $y\dfrac{dz}{dt}-z\dfrac{dy}{dt}=lH,\ z\dfrac{dx}{dt}-x\dfrac{dz}{dt}=mH,\ x\dfrac{dy}{dt}-y\dfrac{dx}{dt}=nH.$

Also, since the velocity perpendicular to the plane of the instantaneous orbit is always zero,

$$\frac{d\zeta}{dt}=\frac{d\zeta}{di}\,\frac{di}{dt}+\frac{d\zeta}{d\Omega}\,\frac{d\Omega}{dt}=0\,\ldots\ldots\ldots(6);$$

and $$\frac{d\zeta}{d\Omega}=l\frac{dx}{d\Omega}+m\frac{dy}{d\Omega}+n\frac{dz}{d\Omega}=-\left(x\frac{dl}{d\Omega}+y\frac{dm}{d\Omega}+z\frac{dn}{d\Omega}\right)$$

$$=-(x\cos\Omega+y\sin\Omega)\sin i\ldots(7),$$

$$\frac{d\zeta}{di}=-\left(x\frac{dl}{di}+y\frac{dm}{di}+z\frac{dn}{di}\right)=-\frac{x}{n}\left(n\frac{dl}{di}-l\frac{dn}{di}\right)-\frac{y}{n}\left(n\frac{dm}{di}-m\frac{dn}{di}\right)$$

$$=-(x\sin\Omega-y\cos\Omega)\sec i\ldots(8).$$

Now, from (5) and (7),

$$\frac{dn}{dt}=-\sin i\,\frac{di}{dt}=-\frac{S}{H}(x\cos\Omega+y\sin\Omega)\sin i,$$

$$\therefore\ \frac{di}{dt}=\frac{S}{H}(x\cos\Omega+y\sin\Omega)=-\frac{1}{H\sin i}\,\frac{dR}{d\zeta}\cdot\frac{d\zeta}{d\Omega}.$$

[redacted]

The angle ϑ is sometimes called the longitude of the node in the orbit. When it is determined, the position of the fixed line is known, and then the planet's place can be found by measuring the angles from it.

The better method is at once to measure the angles from the line of nodes, or, which gives a simpler result, to measure the longitude of perihelion from the line of nodes, and the epoch, which will in this case be the value of the mean anomaly (instead of the mean longitude) when $t = 0$, from perihelion. Let ω be the longitude of perihelion, and c the epoch thus measured.

If now Ω is to be made to vary alone, the orbit must turn about an axis perpendicular to the plane xy, so as to preserve i, ω, and c unchanged. This motion is the same as rotation about an axis in its plane perpendicular to the line of nodes through an angle $d\Omega \sin i$, combined with rotation about the normal to its plane through an angle $d\Omega \cos i$.

Now $$\frac{dR}{d\Omega} = \frac{dR}{d\xi}\,\frac{d\xi}{d\Omega} + \frac{dR}{d\eta}\,\frac{d\eta}{d\Omega} + \frac{dR}{d\zeta}\,\frac{d\zeta}{d\Omega},$$

and $\left(\frac{dR}{d\xi}\,\frac{d\xi}{d\Omega} + \frac{dR}{d\eta}\,\frac{d\eta}{d\Omega}\right) d\Omega$ is the variation of R in consequence of the rotation of the orbit through the angle $d\Omega \cos i$ about the normal to its plane. But the only element affected by rotation about the normal is ω; therefore the change of R by rotation through the angle $d\Omega \cos i$ is $= \frac{dR}{d\omega}\, d\Omega \cos i$;

therefore $$\frac{dR}{d\xi}\,\frac{d\xi}{d\Omega} + \frac{dR}{d\eta}\,\frac{d\eta}{d\Omega} = \frac{dR}{d\omega}\cos i;$$

and therefore $$\frac{dR}{d\zeta}\,\frac{d\zeta}{d\Omega} = \frac{dR}{d\Omega} - \frac{dR}{d\omega}\cos i.$$

The value of $\frac{dR}{d\zeta}\,\frac{d\zeta}{di}$ is the same as before, since the motion of the plane of the orbit, when i alone varies, is the same. The expressions (A) in this case will therefore become

$$\left.\begin{aligned} \frac{di}{dt} &= -\frac{1}{H\sin i}\,\frac{dR}{d\Omega} + \frac{\cos i}{H\sin i}\,\frac{dR}{d\omega} \\ \frac{d\Omega}{dt} &= \frac{1}{H\sin i}\,\frac{dR}{di} \end{aligned}\right\} \quad \ldots\ldots (b).$$

If the epoch be measured from the line of nodes, it is at once

[redacted]

Throughout, H may be replaced by the equivalent expressions $na^2\sqrt{(1-e^2)}$ or $\sqrt{\{\mu a\,(1-e^2)\}}$. The complete series of the variations in the different cases is subjoined, for facility of reference:

$$\frac{da}{dt} = \frac{2}{na}\,\frac{dR}{d\epsilon},$$

$$\frac{de}{dt} = \frac{1-e^2}{na^2e}\cdot\frac{dR}{d\epsilon} - \frac{\sqrt{(1-e^2)}}{na^2e}\left(\frac{dR}{d\epsilon} + \frac{dR}{d\varpi}\right),$$

$$\frac{d\varpi}{dt} = \frac{\sqrt{(1-e^2)}}{na^2e}\cdot\frac{dR}{de},$$

$$\frac{d\epsilon}{dt} = -\frac{2}{na}\,\frac{dR}{da} + \frac{1}{na^2e}\left\{\sqrt{(1-e^2)} - (1-e^2)\right\}\frac{dR}{de},$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{(1-e^2)}\,.\,\sin i}\,\frac{dR}{di},$$

$$\frac{di}{dt} = -\frac{1}{na^2\sqrt{(1-e^2)}\,.\,\sin i}\,\frac{dR}{d\Omega},$$

when the angles are measured from a fixed line in the plane. When they are measured from the line of nodes, the only change in the expressions for $\frac{da}{dt}$ and $\frac{de}{dt}$ is to put ε and ω for ϵ and ϖ. Instead of $\frac{d\epsilon}{dt}$ and $\frac{d\varpi}{dt}$, we have

$$\frac{d\omega}{dt} = \frac{\sqrt{(1-e^2)}}{na^2e}\,\frac{dR}{de} - \frac{\cos i}{na^2\sqrt{(1-e^2)}\,.\,\sin i}\,\frac{dR}{di},$$

$$\frac{d\varepsilon}{dt} = -\frac{2}{na}\,\frac{dR}{da}$$

$$+\frac{1}{na^2e}\left\{\sqrt{(1-e^2)} - (1-e^2)\right\}\frac{dR}{de} - \frac{\cos i}{na^2\sqrt{(1-e^2)}\sin i}\,\frac{dR}{di},$$

and the expression for $\frac{di}{dt}$ becomes

$$= -\frac{1}{na^2\sqrt{(1-e^2)}\sin i}\,\frac{dR}{d\Omega} + \frac{\cos i}{na^2\sqrt{(1-e^2)}\sin i}\left(\frac{dR}{d\omega} + \frac{dR}{d\varepsilon}\right).$$

The change in this expression, it must be remembered, arises, not from any change in the *value* of $\frac{di}{dt}$, but from the different manner in which Ω is involved in R.

[redacted]

The symbolic sum of any two lines is found to be *independent of their order*, in virtue of the same interpretation; so that the equation

$$\mathrm{FE} + \mathrm{HG} = \mathrm{HG} + \mathrm{FE} \quad \ldots\ldots\ldots\ldots (11),$$

is true, in the present system, *not as an independent definition*, but rather as one of the modes of *symbolically expressing that elementary theory of geometry*, (*Euclid*, I. 33), on which was founded the rule for deducing, from any equation (2) between lines, the *alternate* equation (5). For if we assume, as we may, that three points A, B, C, have been so chosen as to satisfy the equations $\mathrm{FE} = \mathrm{BA}$, $\mathrm{HG} = \mathrm{CA}$; and that a fourth point D is chosen so as to satisfy the equation $\mathrm{DC} = \mathrm{BA}$; the same points will then, by the theorem just referred to, satisfy also the equation $\mathrm{DB} = \mathrm{CA}$; and the truth of the formula (11) will be proved, by observing that each of the two symbols which are equated in that formula is equal to the symbol DA, in virtue of the definition (7) of $+$, without any new definition: since

$$\mathrm{FE} + \mathrm{HG} = \mathrm{DC} + \mathrm{CA} = \mathrm{DA} = \mathrm{DB} + \mathrm{BA} = \mathrm{HG} + \mathrm{FE}.$$

A like result is easily shown to hold good, for any number of summands; thus

$$\mathrm{FE} + \mathrm{HG} + \mathrm{KI} = \mathrm{KI} + \mathrm{HG} + \mathrm{FE} \quad \ldots\ldots (12);$$

since the first member of this last equation may be put successively under the forms

$$(\mathrm{FE} + \mathrm{HG}) + \mathrm{KI},\ \mathrm{KI} + (\mathrm{FE} + \mathrm{HG}),\ \mathrm{KI} + (\mathrm{HG} + \mathrm{FE}),$$

and finally under the form of the second member; the stages of this successive transformation of symbols admitting easily of geometrical interpretations: and similarly in other cases. *Addition of lines in space* is therefore generally (as Mr. Warren has shewn it to be for lines in a single plane) a *commutative operation;* in the sense that the summands may interchange their places, without the sum being changed. It is also an *associative* operation, in the sense that any number of successive summands may be associated into one group, and collected into one partial sum (denoted by enclosing these summands in parentheses); and that then this partial sum may be added, as a single summand, to the rest: thus

$$(\mathrm{KI} + \mathrm{HG}) + \mathrm{FE} = \mathrm{KI} + (\mathrm{HG} + \mathrm{FE}) = \mathrm{KI} + \mathrm{HG} + \mathrm{FE} \ldots (13).$$

On the mark $-$.

4. The equation* $\quad \mathrm{CA} - \mathrm{BA} = \mathrm{CB} \quad \ldots\ldots\ldots\ldots (14)$

* On the plan mentioned in some former notes, this equation would take the form

$$(\mathrm{C} - \mathrm{A}) - (\mathrm{B} - \mathrm{A}) = \mathrm{C} - \mathrm{B}.$$

[redacted]

as the small roman letters a, b, &c., with or without accents, as symbols of lines, instead of binary combinations of the roman capitals, in cases where the lines which are compared are not supposed to have necessarily any common point, and generally when the *situations* of lines are disregarded, but not their lengths nor their directions. Thus we shall have, instead of (11) and (12), (13), (15) and (16), these other formulæ of the present Symbolical Geometry, which agree in all respect with those used in Symbolical Algebra:

$$\mathrm{a} + \mathrm{b} = \mathrm{b} + \mathrm{a}, \quad \mathrm{a} + \mathrm{b} + \mathrm{c} = \mathrm{c} + \mathrm{b} + \mathrm{a} \ldots\ldots (19);$$

$$(\mathrm{c} + \mathrm{b}) + \mathrm{a} = \mathrm{c} + (\mathrm{b} + \mathrm{a}) = \mathrm{c} + \mathrm{b} + \mathrm{a} \ \ldots\ldots (20);$$

$$(\mathrm{b} - \mathrm{a}) + \mathrm{a} = \mathrm{b}, \quad (\mathrm{b} + \mathrm{a}) - \mathrm{a} = \mathrm{b} \ldots\ldots\ldots (21);$$

and because the isolated but *affected* symbols $+\mathrm{a}$, $-\mathrm{a}$, may denote, by (18), the line a itself, and the opposite of that line, we have also here the usual *rule of the signs*,

$$+(+\mathrm{a}) = -(-\mathrm{a}) = +\mathrm{a}, \quad +(-\mathrm{a}) = -(+\mathrm{a}) = -\mathrm{a} \ldots (22).$$

Introduction of the marks × *and* ÷.

6. Continuing to denote lines by letters, the formula

$$(\mathrm{b} \div \mathrm{a}) \times \mathrm{a} = \mathrm{b} \ldots\ldots\ldots\ldots\ldots (23),$$

which is, for the relation between multiplication and division, what the first of the two formulæ (21) is for the relation between addition and subtraction, will be true, in the most elementary sense of the multiplication of a length by a number, for the case when the line b is the sum of several summands, each equal to the line a, and when the number of those summands is denoted by the quotient $\mathrm{b} \div \mathrm{a}$. And we shall now, for the purposes of symbolical generality, *extend* this formula (23), so as to make it be valid, *by definition, whatever two lines* may be denoted by a and b. The formula will then *express nothing respecting those lines* themselves, which can serve to distinguish them from any other lines in space; but will furnish a *symbolic condition,* which we must satisfy by the *general interpretation* of a *geometrical quotient,* and of the *operation of multiplying a line* by such a quotient.

To make such general interpretation consistent with the particular case where a quotient becomes a *quotity,* we are led to write

$$\mathrm{a} \div \mathrm{a} = 1, \quad (\mathrm{a} + \mathrm{a}) \div \mathrm{a} = 2, \quad \&\mathrm{c}. \ldots\ldots (24),$$

and conversely

$$1 \times \mathrm{a} = \mathrm{a}, \quad 2 \times \mathrm{a} = \mathrm{a} + \mathrm{a}, \quad \&\mathrm{c}. \ldots\ldots\ldots (25);$$

[redacted]

line (b) parallel to the given line (a), the direction of the one being similar or opposite to that of the other, according as the number is positive or negative, while the length of the new line bears to the length of the given line a ratio which is marked by the same given number. So that if A_0 A_1 A_a denote any three points on one common axis of rectilinear progression, which are related to each other, upon that axis, as to their order and their intervals, in the same manner as the three scalar numbers 0, 1, *a*, regarded as ordinals, are related to each other on the scale of numerical progression from $-\infty$ to $+\infty$, then the equations

$$A_aA_0 \div A_1A_0 = a, \quad a \times A_1A_0 = A_aA_0, \ldots\ldots (35)$$

will be true by the foregoing interpretations.

It is easy to see that this mode of interpreting a quotient of parallel lines renders the formulæ (26) (27) (28) (29) consistent with the received rules for performing the operations $+ - \times \div$ on what are called real numbers, whether they be positive or negative, and whether commensurable or incommensurable; or rather reproduces those rules as consequences of those formulæ.

On Vectors, and Geometrical Quotients in general.

7. The other chief relation of directions of lines in space, besides parallelism, is perpendicularity; which it is not unusual to denote by writing the mark $\perp$ between the symbols of two perpendicular lines. And the other chief class of geometrical quotients which it is important to study, as preparatory to a general theory of such quotients, is the class in which the dividend is a line perpendicular to the divisor. A quotient of this latter class we shall call a VECTOR, to mark its connection (which is closer than that of a *scalar*) with the conception of *space*, and for other reasons which will afterwards appear: and if we agree to denote, for the present, such vector quotients (of perpendicular lines) by small Greek letters, in contrast to the scalar class of quotients (of parallel lines) which we have proposed to denote by small italic letters, we shall then have generally two equations of the forms

$$c \div a = \alpha, \quad c = \alpha \times a, \quad \text{if } c \perp a \ldots\ldots\ldots (36).$$

Any line e may be put under the form c + b, in which b $\parallel$ a, and c $\perp$ a; a *general geometrical quotient* may therefore, by (26) (34) (36), be considered as the *symbolic sum of a scalar and a vector*, zero being regarded as a common limit of quotients of these two classes; and consequently, if we

[redacted]

turn, right-handedly, through a right angle, in order to attain the original direction of the dividend line c. A line drawn in the direction of this *axis of* (what is here regarded as) *positive rotation,* and having its length in the same ratio to some assumed *unit* of length as the length of the dividend to that of the divisor, may be called the INDEX of the vector. We shall thus be led to substitute, for any equation between two vector quotients, an equation between two lines, namely between their indices; for if we define that two vector quotients, such as $c \div a$ and $c' \div a'$ if $c \perp a$ and $c' \perp a'$, are *equal* when they have *equal indices,* we shall satisfy all conditions of symbolical equality, of the kinds already considered in connection with other definitions; we shall also be able to say that in every case of two such equal quotients, the two dividend lines (c and c′) bear to their own divisor lines (a and a′), respectively, one common ratio of lengths, and one common relation of directions. We shall thus also, by (23), be able to *interpret the multiplication* of any given line a′ by any given vector $c \div a$, *provided that the one is perpendicular to the index of the other,* as the operation of deducing from a′ another line c′, by altering (generally) its length in a given ratio, and by turning (always) its direction round a given axis of rotation, namely round the index of the vector, right-handedly, through a right angle. And we can now *interpret an equation between two general geometrical quotients,* such as

$$e' \div a' = e \div a \ldots\ldots\ldots\ldots\ldots (40),$$

as being equivalent to a *system of two separate equations,* one between the scalar and another between the vector parts, namely the two following:

$$S(e' \div a') = S(e \div a);\ V(e' \div a') = V(e \div a) \ldots (41);$$

of which each separately is to be interpreted on the principles already laid down; and which are easily seen (by considerations of similar triangles) to imply, when taken jointly, that the length of e′ is to that of a′ in the same ratio as the length of e to that of a; and also that the same rotation, round the index of either of the two equal vectors, which would cause the direction of a to attain the original direction of e, would also bring the direction of a′ into that originally occupied by e′. At the same time we see how to interpret the operation of multiplying any given line a′ by any given geometrical quotient $e \div a$ of two other lines, *whenever the three given lines* a, e, a′, *are parallel to one common plane;* namely as being the complex operation of altering (generally) a given length in a given ratio, and of turning a given line

reciprocally proportional to the perpendicular on the tangent plane at the corresponding point on the other.

For if r, p, be the radius vector and perpendicular on the tangent plane at one point, and R, P, at the other, xyz, XYZ the coordinates of these points respectively,

$$r^2 = x^2 + y^2 + z^2$$

$$= \frac{a^2}{A^2} X^2 + \frac{b^2}{B^2} Y^2 + \frac{c^2}{C^2} Z^2 = \frac{X^2}{A^4} + \frac{Y^2}{B^4} + \frac{Z^2}{C^4} = \frac{1}{P^2}.$$

Similarly $R = \frac{1}{p}$, and the cosines of the angles which r makes with the axes are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$, or $\frac{PX}{A^2}, \frac{PY}{B^2}, \frac{PZ}{C^2}$, which are the cosines of the angles made by P with the axes. Hence it appears that r is coincident with and reciprocally proportional to P.

Prop. I. If p be the perpendicular from the centre of an ellipsoid on the tangent plane, θ and ϕ the angles which determine its position, and a, b, c the semiaxes, the element of the surface is $\frac{a^2 b^2 c^2}{p^4} \cdot \sin\theta \, d\theta \, d\phi$.*

For, let $d\alpha$ represent this element, and ds the solid sector of which it is the base, then $d\alpha = \frac{3ds}{p}$. Now let dS be the corresponding sector of the reciprocal surface, and (Lemma 2) $ds = \frac{abc}{ABC} \cdot dS = a^2 b^2 c^2 \, dS$; and since

$$dS = \tfrac{1}{3} R^3 \sin\theta \, d\theta \, d\phi = \tfrac{1}{3} \frac{\sin\theta \, d\theta \, d\phi}{p^3},$$

$$d\alpha = \frac{a^2 b^2 c^2}{p^4} \cdot \sin\theta \, d\theta \, d\phi. \quad \text{Q. E. D.}$$

Prop. II. To integrate the above expression for the ellipsoid.

Assume
$$m^2 = a^2 \sin^2\theta + c^2 \cos^2\theta,$$
$$n^2 = b^2 \sin^2\theta + c^2 \cos^2\theta;$$

then $p^2 = a^2 \sin^2\theta \cos^2\phi + b^2 \sin^2\theta \sin^2\phi + c^2 \cos^2\theta$

$$= m^2 \cos^2\phi + n^2 \sin^2\phi,$$

* Since arriving at the above expression I found that it had been previously given, with a different demonstration, by Jacobi. He has, however, only applied it to the ellipsoid, and his memoir has very little in common with the present. The theorem in Prop. IX. of the present paper is, I believe, the first attempt to compare different portions of the ellipsoidal surface.

PROP. III. The portion of the superficial area of the ellipsoid included between any two of the curves described in the foregoing proposition, may be expressed by means of arcs of the focal conics of the reciprocal ellipsoid.

We have seen that if dS be the elementary area included between two consecutive curves of the foregoing species,

$$dS = \pi a^2 b^2 c^2 \sin\theta\, d\theta \left(\frac{1}{nm^3} + \frac{1}{n^3 m}\right).$$

Hence, if any two curves of this species be described, for one of which the value of θ is α, and for the other α', the value of the superficial area which they include is

$$S = \pi a^2 b^2 c^2 (I + I'), \text{ where}$$

$$I = \int_\alpha^{\alpha'} \frac{\sin\theta\, d\theta}{(b^2 \sin^2\theta + c^2\cos^2\theta)^{\frac{3}{2}} (a^2\sin^2\theta + c^2\cos^2\theta)^{\frac{1}{2}}},$$

$$I' = \int_\alpha^{\alpha'} \frac{\sin\theta\, d\theta}{(a^2 \sin^2\theta + c^2\cos^2\theta)^{\frac{3}{2}} (b^2\sin^2\theta + c^2\cos^2\theta)^{\frac{1}{2}}},$$

The first of these integrals denotes the arc of an ellipse, and the second that of a hyperbola, it being supposed that $a > b > c$.

Assume $a^2 - c^2 = a^2e^2$, $b^2 - c^2 = b^2e'^2$, $e'\cos\theta = \sin u$,

and $I = \dfrac{1}{e'ab^3} \cdot \displaystyle\int \frac{\sec^2 u\, du}{\sqrt{\left(1 - \frac{e^2}{e'^2} \cdot \sin^2 u\right)}}$, or putting $x = \tan u$,

$$I = \frac{1}{e'ab^3} \cdot \int \frac{dx\, \sqrt{(1 + x^2)}}{\sqrt{\left\{1 - \left(\frac{e^2}{e'^2} - 1\right) \cdot x^2\right\}}},$$

the limits of x being $\dfrac{e'\cos\alpha}{\sqrt{(1 - e'^2\cos^2\alpha)}}$, $\dfrac{e'\cos\alpha'}{\sqrt{(1 - e'^2\cos^2\alpha')}}$.

Now let σ be the arc of an ellipse whose semiaxes are A and B, and it is known that $d\sigma = \dfrac{dy\, \sqrt{\{B^4 + (A^2 - B^2)y^2\}}}{B\sqrt{(B^2 - y^2)}}$,

or if we put $A^2 - B^2 = \dfrac{B^2}{K^2}$, and $y = BKx$,

$$d\sigma = BKdx\, \sqrt{\left(\frac{1 + x^2}{1 - K^2x^2}\right)}.$$

bounded by four lines of curvature; but he has not found that his expression may be reduced to a single definite integral, which Mr. Jellett's investigations shew to be possible. See Legendre, *Traité des Fonctions Elliptiques*, vol. I. p. 350. Also Catalan, *Sur la Transformation des Variables dans les Intégrales Multiples. Mémoires Couronnés par l'Académie de Bruxelles*, 1839-40.]

To make this coincide with the foregoing expression for I, assume $\frac{e^2}{e'^2} - 1 = K^2$, and we shall find $\frac{A^2}{B^2} = \frac{\frac{1}{c^2} - \frac{1}{a^2}}{\frac{1}{b^2} - \frac{1}{a^2}}$. The ellipse therefore by means of which I is represented is similar to the focal ellipse of the reciprocal surface; and since there is nothing to determine its absolute magnitude, it may be taken to be the focal ellipse itself. If then σ be the arc of this ellipse, whose ordinates measured parallel to the minor axis are $y = \frac{BKe' \cos \alpha}{\sqrt{(1 - e'^2 \cos^2 \alpha)}}$, and $y' = \frac{BKe' \cos \alpha'}{\sqrt{(1 - e'^2 \cos^2 \alpha')}}$, we shall have $I = \frac{\sigma}{BKe'ab^3} = \frac{\sigma}{ab^3 B \sqrt{(e^2 - e'^2)}}$.

Now let a', b', c', be the semiaxes of the reciprocal ellipsoid, and it is evident that $e^2 - e'^2 = \frac{b'^2 - a'^2}{c'^2} = \frac{B^2}{C'^2}$,

therefore
$$I = \frac{c'\sigma}{ab^3 B^2}.$$

By using a reduction precisely similar, we find $I' = \frac{c' \sigma'}{a^3 b B^2}$, where σ' is an arc of the focal hyperbola, whose extreme ordinates are $y = \frac{BK'e \cos \alpha}{\sqrt{(1 - e^2)} \cos^2 \alpha}$ and $y' = \frac{BK'e \cos \alpha'}{\sqrt{(1 - e^2 \cos^2 \alpha')}}$. Hence we find ultimately for the required superficial area the expression
$$S = \frac{\pi c^2 c'}{B^2} \cdot \left(\frac{a}{b}\ \sigma + \frac{b}{a}\ \sigma'\right),$$
or if we assume the reciprocal ellipsoid such that
$$aa' = bb' = cc' = B^2,$$
$$S = \pi c \,.\left(\frac{a}{b} \,.\, \sigma + \frac{b}{a} \,.\, \sigma'\right).$$

PROP. IV. To construct geometrically the limits of the integrals I and I'.

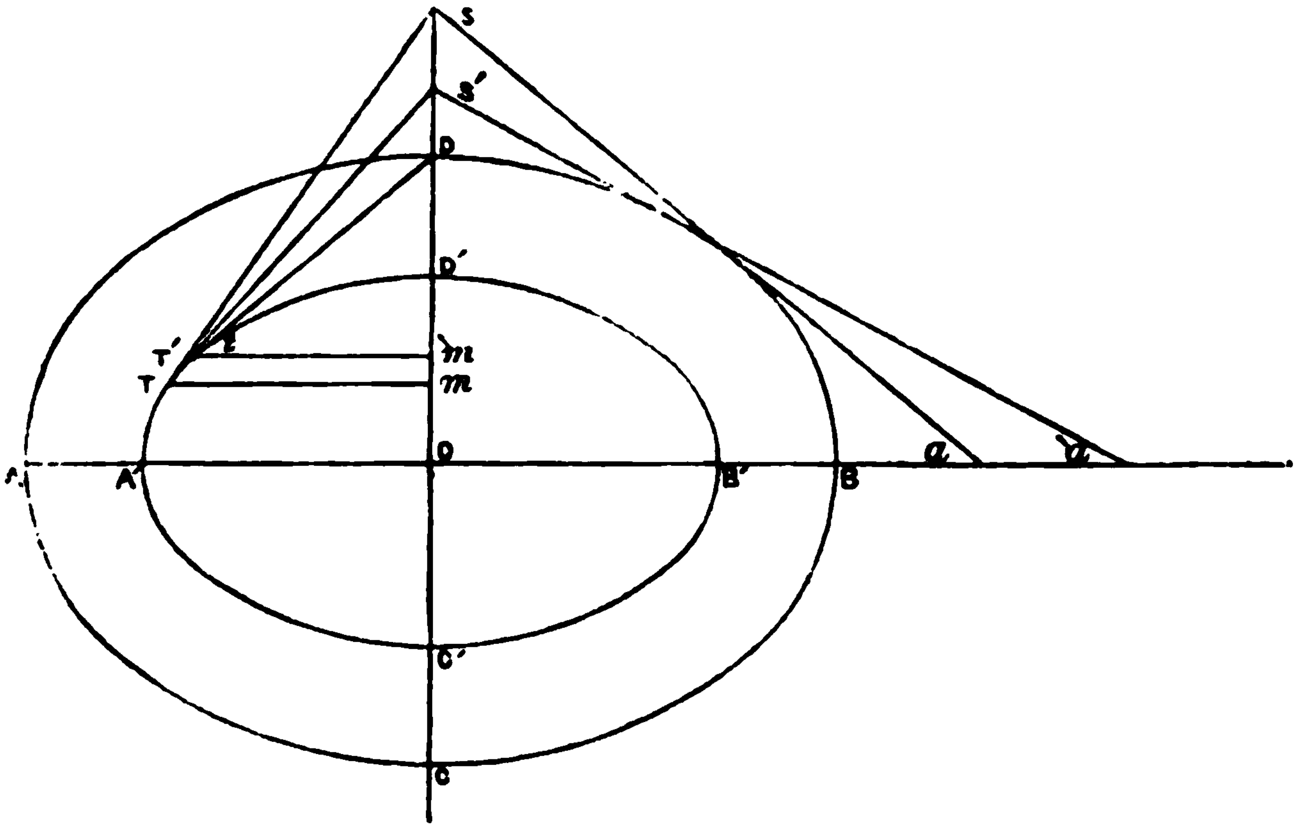
As the expressions for the limiting values of the ordinates are precisely similar for the two integrals, it will be sufficient to consider one of them.

We have seen that if y be one of the limiting ordinates, $y = \frac{BKe' \cos \alpha}{\sqrt{(1 - e'^2 \cos^2 \alpha)}}$, or, substituting for e' and K,
$$y = \frac{B^2 \cos \alpha}{\sqrt{(c'^2 \sin^2 \alpha + b'^2 \cos^2 \alpha)}} = \frac{B^2}{p \sec \alpha},$$

where p is the perpendicular drawn from the centre of the ellipse whose axes are c', b', (*i.e.* the principal section of the reciprocal ellipsoid which contains its greatest and mean axis) on the tangent, which makes the angle α with the axis c'.

Similarly $$y' = \frac{B^2}{p'.\sec\alpha'}.$$

Hence we derive the following construction. Let $ACBD$ be the section of the reciprocal ellipsoid containing the greatest



and mean axis, $A'C'B'D'$ the focal ellipse of the same surface. Draw two tangents to the outer ellipse, making with the axis AB the angles α, α', and from the points s, s', where they cut the other axis, draw tangents to the inner ellipse. TT' is the arc which has become denoted by σ. For

$$Om = \frac{OD'^2}{OS} = \frac{B'^2}{p\sec\alpha}$$

equals one of the limiting ordinates. The other is of course Om'.

The same construction precisely applies to the focal hyperbola: if then t and t' be the points of contact of the similarly drawn tangents to it, the surface of the ellipsoidal belt will be

$$S = \pi c.\left(\frac{a}{b}\,TT' + \frac{b}{a}\,tt'\right).$$

For the semiellipsoid, $\alpha = \frac{\pi}{2}$, $\alpha' = 0$; therefore the point S will go to infinity, S' will coincide with D, and the arc TT' will become $A't$. Similar reductions hold for the hyperbolic arc tt'.

PROP. V. The superficial area of the hyperboloids may, to a certain extent, be expressed in the same manner.

In the hyperboloid of one sheet, if θ be measured from the imaginary axis, we shall find the preceding investigation strictly applicable for all values of θ between $\frac{\pi}{2}$ and $\tan^{-1}\frac{c}{b}$. For values less than this, the integrals I and I' become imaginary, which is explained by observing that the cones whose intersections with the surface give the bounding curves, will in this case fall partly outside the surface. In the hyperboloid of two sheets it is necessary to measure θ from the real axis, and in this I and I' continue real from

$$\theta = 0 \text{ to } \theta = \tan^{-1}\frac{c}{a};$$

after this they become imaginary. The explanation of this is the same as for the hyperboloid of one sheet.

PROP. VI. To find what expression is to be substituted for $a^2 b^2 c^2 \dfrac{\sin\theta \, d\theta \, d\phi}{p^4}$ in the case of either of the paraboloids.

Adopting the usual mode of deriving the properties of the paraboloid from those of the ellipsoid or hyperboloid, we shall put $a^2 = mc$, $b^2 = nc$, and then make c infinite. Performing these operations we shall find

$$\frac{a^2 b^2 c^2}{p^4} = \frac{mnc^4}{(mc\sin^2\theta\cos^2\phi + nc\sin^2\theta\sin^2\phi + c^2\cos^2\theta)^2}$$

$$= \frac{mn}{\left(\dfrac{m\sin^2\theta\cos^2\phi + n\sin^2\theta\sin^2\phi}{c} + \cos^2\theta\right)^2} = \frac{mn}{\cos^4\theta}.$$

Hence we have $dS = mn.\dfrac{\sin\theta d\theta \,.\, d\phi}{\cos^4\theta}$

PROP. VII. To integrate the above expression and construct the bounding curves.

$$S = mn\int_a^{a'}\frac{\sin\theta d\theta}{\cos^4\theta}\int_0^{2\pi} d\phi = \frac{2\pi mn}{3}(\sec^3 a' - \sec^3 a).$$

The equations of the bounding curves are as before, $\theta = a$, $\theta = a'$: and since the equation of the tangent plane to the paraboloid is

$$z - z' = \frac{x'}{m}(x - x') + \frac{y'}{n}(y - y'),$$

if x, y be two of the coordinates of a point on one of these curves (the axis of z being the axis of the paraboloid), we shall have $\frac{x^2}{m^2} + \frac{y^2}{n^2} = \tan^2 \alpha$, and $\frac{x^2}{m^2} + \frac{y^2}{n^2} = \tan^2 \alpha'$. Hence the curves may be constructed as follows. In any plane perpendicular to the axis of the paraboloid, describe two ellipses whose axes are in the planes of the principal sections of the paraboloid and proportional to their parameters, and on these ellipses as bases erect two cylinders whose generatrices are parallel to the axis of the paraboloid. These cylinders will cut the surface in the required curves.

Prop. VIII. The paraboloidal belt intercepted between any two of the curves described in the foregoing proposition, is proportional to the difference between the radii of curvature of either of the principal sections at the points where they intersect the bounding curves.

It appears from the preceding proposition that

$$S = \frac{2\pi mn}{3} \cdot (\sec^3 \alpha' - \sec^3 \alpha),$$

α, α' being the angles made with the axis by the normal to the surface at any point on the bounding curves. Let R be the radius of curvature, and N the normal to the principal section whose semi-parameter is m at the point where it intersects the first of the bounding curves; then, since N is also normal to the surface, $\sec^3 \alpha = \frac{N^3}{m^3} = \frac{R}{m}$, $\left(\text{since } R = \frac{N^3}{m^2}\right)$. Similarly $\sec^3 \alpha' = \frac{R'}{m}$; therefore $S = \frac{2\pi n}{3} (R' - R)$. Q.E.D.

It is evident that if ρ, ρ' be the similar radii of curvature for the other principal section we shall have $S = \frac{2\pi m}{3} (\rho' - \rho)$. It appears also that if, with the same parameters and with the same principal planes, there be constructed two paraboloids, one elliptic, the other hyperbolic; the cylinders described in Prop. VII. will intercept on them portions whose superficial areas are the same.

Prop. IX. Let three curves be described on the surface of an ellipsoid along the first of which the perpendicular to the tangent plane makes with the axis of z the constant angle γ, along the second β with the axis of y, and along the third α with the axis of x, and let these angles be connected by the

equations $\dfrac{\tan\alpha}{a} = \dfrac{\tan\beta}{b} = \dfrac{\tan\gamma}{c}$;* then if A_3, A_2, A_1 be the included portions of the ellipsoidal surface, we shall have

$$\frac{A_3 - A_2}{a^2} + \frac{A_1 - A_3}{b^2} + \frac{A_2 - A_1}{c^2} = 0.$$

It appears from Prop. 3, that

$$dA_3 = \frac{\pi a^2 b^2 c^2 . \sin\theta d\theta}{\sqrt{\{(a^2\sin^2\theta + c^2\cos^2\theta) . (b^2\sin^2\theta + c^2\cos^2\theta\}}} . \left\{\frac{1}{a^2\sin^2\theta + c^2\cos^2\theta} + \frac{1}{b^2\sin^2\theta + c^2\cos^2\theta}\right\}:$$

And in the same way, if we had supposed the angle θ to be measured from the axis of y, we should have had

$$dA_2 = \frac{\pi a^2 b^2 c^2 \sin\theta d\theta}{\sqrt{\{(a^2\sin^2\theta + b^2\cos^2\theta)(c^2\sin^2\theta + b^2\cos^2\theta)\}}} . \left(\frac{1}{a^2\sin^2\theta + b^2\cos\theta} + \frac{1}{c^2\sin^2\theta + b^2\cos^2\theta}\right).$$

and by measuring θ from the axis of x we should have a similar value for dA_1. Now if in the values of dA_1, dA_2, dA_3, respectively, for $\tan\theta$ we substitute ax, bx, cx, it is evident that the limits of integration with regard to x will be the same for all; and it is easy to see that the values of dA_1, dA_2, dA_3, may be put under the following forms,

$$dA_1 = \pi b^2 c^2 \{2 + (b^2 + c^2) x^2\} (1 + a^2 x^2)^2 dL \ldots .(1),$$
$$dA_2 = \pi a^2 c^2 \{2 + (a^2 + c^2) x^2\} (1 + b^2 x^2)^2 dL \ldots .(2),$$
$$dA_3 = \pi a^2 b^2 \{2 + (a^2 + b^2) x^2\} (1 + c^2 x^2)^2 dL \ldots .(3),$$

where $$dL = \frac{xdx}{(1 + a^2x^2)^{\frac{3}{2}} . (1 + b^2x^2)^{\frac{3}{2}} (1 + c^2x^2)^{\frac{3}{2}}}.$$

Multiply equation (1) by $(b^2 - c^2) a^2$, equation (2) by $(c^2 - a^2) b^2$, and equation (3) by $(a^2 - b^2) c^2$, and add them, and it is easy to see that the right-hand member of the new equation will vanish, hence

$$(b^2 - c^2) a^2 dA_1 + (c^2 - a^2) . b^2 dA_2 + (a^2 - b^2) c^2 dA_3 = 0,$$

a result which may be put under the form

$$\frac{d(A_1 - A_2)}{a^2} + \frac{d(A_1 - A_3)}{b^2} + \frac{d(A_2 - A_1)}{c^2} = 0,$$

* A relation analogous to this subsists between the perpendiculars on the tangents at the extremities of the elliptical arcs used in Fagnani's theorem, for if α be the angle made with the axis of x by the perpendicular corresponding to the arc which terminates at the extremity of that axis (a) and β the similar angle for the axis b, we shall have $\dfrac{\tan\alpha}{a} = \dfrac{\tan\beta}{b}$, as is easily seen.

[redacted]

And since A_1, A_2, A_3, all begin together, the proposition is evident. If instead of supposing A_1, A_2, A_3, to be bounded, each by a single curve, we conceive each of these letters to denote the space included between two such curves, the same theorem holds, provided that the curves of the second series are connected by the same equations as those of the first.

ON THE POLAR EQUATION TO A CHORD OF A CONIC SECTION.

By the Rev. Percival Frost, M.A., St. John's College.

In a previous number of the *Mathematical Journal* having noticed a form of the polar equation to the tangent to a conic section, I think that the corresponding equation to the chord, which appears nearly in the same form, may be thought worthy of notice by some of the readers of the Journal.

Let the equation to the conic section be

$$\frac{c}{r} = 1 + e \cos \theta,$$

$\alpha + \beta$, $\alpha - \beta$ the values of θ which correspond to the points of intersection of the chord and conic section, and

$$\frac{c}{r} = m \cos \theta + n \sin \theta$$

the equation to the chord.

At the points of intersection we obtain by equating the sides of the equations

$$(m - e) \cos (\alpha - \beta) + n \sin (\alpha - \beta) = 1,$$

$$(m - e) \cos (\alpha + \beta) + n \sin (\alpha + \beta) = 1.$$

Hence $\quad (m - e) \cos \alpha \cos \beta + n \sin \alpha \cos \beta = 1,$

and $\quad (m - e) \sin \alpha \sin \beta - n \cos \alpha \sin \beta = 0;$

then $\dfrac{m-e}{\cos \alpha} = \dfrac{n}{\sin \alpha} = \dfrac{(m-e) \cos \alpha + n \sin \alpha}{\cos^2 \alpha + \sin^2 \alpha} = \dfrac{\sec \beta}{1} = \sec \beta.$

Therefore the equation to the chord of the conic section is

$$\frac{c}{r} = (e + \sec \beta \cos \alpha) \cos \theta + \sec \beta \sin \alpha \sin \theta$$

$$= \sec \beta \cos (\theta - \alpha) + e \cos \theta.$$

Cor. If $\beta = 0$, we obtain the equation to the tangent at the point $\theta = \alpha$,

$$\frac{c}{r} = \cos (\theta - \alpha) + e \cos \theta.$$

By means of this equation the problems proposed in vol. III. p. 87, may be readily solved. For, since the equation to the chord may be written

$$\frac{c \cos \beta}{r} = \cos (\theta - \alpha) + e \cos \beta \cos \theta,$$

this chord touches the conic section whose eccentricity and latus rectum are $e \cos \beta$ and $2c \cos \beta$, the point of contact being in the line bisecting the angle between the distance; if this angle be constant, the conic section is the envelope of the chords.

If the focal distance corresponding to the angle $\alpha - \beta$ be produced, α', β' the values of α, β, corresponding to the produced focal distance and the other

$$\alpha' - \beta' = \alpha + \beta,$$
$$\alpha' + \beta' = \alpha - \beta + \pi,$$

therefore
$$2\beta' = \pi - 2\beta,$$
$$\beta' = \frac{\pi}{2} - \beta.$$

And the envelope to the corresponding chord has for its equation

$$\frac{c \sin \beta}{r} = \cos (\theta - \alpha') + e \sin \beta \cos \theta,$$

and
$$c^2 = (c \sin \beta)^2 + (c \cos \beta)^2,$$
$$e^2 = (e \sin \beta)^2 + (e \cos \beta)^2,$$

which prove the propositions.

Several problems may be conveniently solved by means of this equation.

Prob. 1. If the angle between two focal distances be bisected by a third which remains fixed in position, the chords joining the extremities of the two focal distances, as they change their position, always pass through a fixed point whose locus is the directrix.

The equation to any chord is

$$\frac{c \cos \beta}{r} = \cos (\theta - \alpha) + e \cos \beta \cos \theta;$$

therefore at the point of intersection with any other, α being constant,
$$\cos (\theta - \alpha) = 0 \quad \ldots\ldots\ldots\ldots\ldots (1),$$

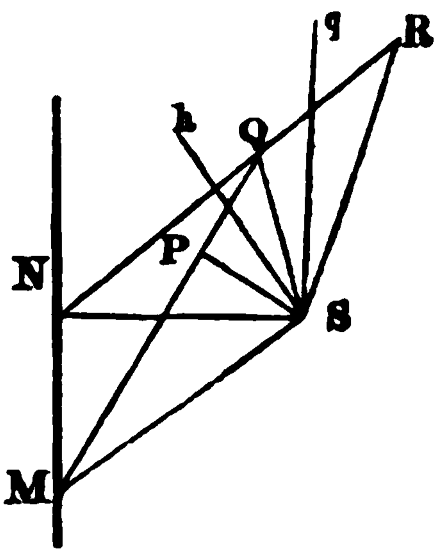
and
$$\frac{c}{r} = e \cos \theta \quad \ldots\ldots\ldots\ldots\ldots (2);$$

therefore, by (1),
$$\theta = \alpha \mp \frac{\pi}{2} \quad \ldots\ldots\ldots\ldots\ldots (3),$$

and by (2) the locus is the directrix (4). Hence the following geometrical construction may be easily performed.

PROB. 2. If three points P, Q, R be a conic section, and the focus be given, to construct the directrix.

Bisect the angles PSQ, QSR, by Sh and Sq. Draw SM and SN perpendicular to Sh, Sq. Produce QP and RQ to meet them in M and N. Join MN, which is the directrix, as appears from (3) and (4) of last problem.



PROB. 3. The locus of the intersection of chords drawn so that β in both is the same, and the difference between the α' constant, is a conic section.

ON THE REDUCTION OF $\dfrac{du}{\sqrt{U}}$, WHEN U IS A FUNCTION OF THE FOURTH ORDER.

By ARTHUR CAYLEY, M.A., Fellow of Trinity College.

IT is well known that the transformation of this differential expression into a similar one, in which the function in the denominator contains only even powers of the corresponding variable, is the first stêp in the process of reducing $\int\frac{du}{\sqrt{U}}$ to elliptic integrals. And, accordingly, the different modes of effecting this have been examined, more or less, by most of those who have written on the subject. The simplest supposition, that adopted by Legendre, and likewise discussed in some detail by Guderman, is that (u) is a fraction, the numerator and denominator of which are linear functions of the new variable. But the theory of this transformation admits of being developed further than it has yet been done, as regards the equation which determines the modulus of the elliptic function. This may be effected most easily as follows.

Suppose

$$U = a + 4bu + 6cu^2 + 4du^3 + eu^4,$$

$$P = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4.$$

Also let $P' = a'x'^4 + 4b'x'^3y' + 6c'x'^2y'^2 + 4d'x'y'^3 + e'y'^4$

be what P becomes after writing

$$x = \lambda x' + \mu y',$$

$$y = \lambda_, x' + \mu_, y' :$$

and let $\quad U' = a' + 4b'u' + 6c'u'^2 + 4d'u'^3 + e'u'^4.$

Suppose, moreover,

$$\left\{\begin{aligned} k &= \lambda\mu_{,} - \lambda_{,}\mu, \\ I &= ae - 4bd + 3c^2, \\ I' &= a'e' - 4b'd' + 3c'^2, \\ J &= ace - ad^2 - eb^2 - c^3 + 2bdc, \\ J' &= a'c'e' - a'd'^2 - e'b'^2 - c'^3 + 2b'd'c'; \end{aligned}\right.$$

we have evidently

$$xdy - ydx = k.(x'dy' - y'dx'),$$

or

$$\frac{xdy - ydx}{P^{\frac{1}{2}}} = k.\frac{x'dy' - y'dx'}{P'^{\frac{1}{2}}}.$$

Or, writing $\quad u = \dfrac{y}{x}, \quad u' = \dfrac{y'}{x'};$

and therefore

$$\frac{xdy - ydx}{P^{\frac{1}{2}}} = \frac{du}{U^{\frac{1}{2}}}, \quad \frac{x'dy' - y'dx'}{P'^{\frac{1}{2}}} = \frac{du'}{U'^{\frac{1}{2}}}$$

$$\frac{du}{U^{\frac{1}{2}}} = k\,\frac{du'}{U'^{\frac{1}{2}}},$$

the equation between u and u' being

$$u = \frac{\lambda + \mu u'}{\lambda_{,} + \mu_{,} u'}.$$

Next, to determine the relations between the coefficients of U and U'. Since P, P' are obtained from each other by linear transformations (*Math. Journal*, vol. IV. p. 208), we have between the coefficients of these functions and of the transforming equations, the relations

$$I' = k^4.I,$$
$$J' = k^6.J;$$

whence also $\quad \dfrac{J'^2}{I'^3} = \dfrac{J^2}{I^3}.$

Suppose now $\quad U' = a'(1 + pu'^2)(1 + qu'^2),$

or $\quad b' = 0, \quad d' = 0, \quad 6c' = a'(p + q), \quad e' = a'pq;$

whence also $\quad I' = \dfrac{a'^2}{12}.(p^2 + q^2 + 14pq),$

$$J' = \frac{a'^3}{216}(p + q).(34pq - p^2 - q^2);$$

[redacted]

$$p^2 + q^2 + 14pq = 12 \,.\, \frac{k^4}{a'^2} I,$$

$$(p + q) . (34pq - p^2 - q^2) = 216 \,.\, \frac{k^6}{a'^3} J;$$

$$\therefore \frac{(p + q)^2 . (34pq - p^2 - q^2)^2}{(p^2 + q^2 + 14pq)^3}$$

$$= \frac{27J^2}{I^3}, \quad \text{whence also} \quad \frac{108pq\,(p - q)^4}{(p^2 + q^2 + 14pq)^3} = 1 - \frac{27J^2}{I^3},$$

which determines the relation between p and q. Also

$$\frac{k}{\sqrt{a'}} = \left(\frac{p^2 + q^2 + 14pq}{12I}\right)^{\frac{1}{4}},$$

so that $\dfrac{du}{\sqrt{U}} = \left(\dfrac{p^2 + q^2 + 14pq}{12I}\right)^{\frac{1}{4}} \dfrac{du'}{\{(1 + pu'^2)(1 + qu'^2)\}^{\frac{1}{2}}}.$

If in particular $p = -1$, writing also $-q$ for q,

$$\frac{du}{\sqrt{U}} = \left(\frac{q^2 + 14q + 1}{12I}\right)^{\frac{1}{4}} \frac{du'}{\{(1 - u'^2)(1 - qu'^2)\}^{\frac{1}{2}}},$$

where $$\frac{108q(1 - q)^4}{(q^2 + 14q + 1)^3} = 1 - \frac{27J^2}{I^3}.$$

Suppose, for shortness,

$$M = \frac{27}{4} \,.\, \frac{1}{\left(1 - \frac{27J^2}{I^3}\right)}, \quad \text{or} \quad \frac{1}{108}\left(1 - \frac{27J^2}{I^3}\right) = \frac{1}{16M},$$

$$(q^2 + 14q + 1)^3 - 16Mq(q - 1)^4 = 0, \quad i.e.$$

$$\left(q + \frac{1}{q} + 14\right)^3 - 16M\left(q^{\frac{1}{2}} - \frac{1}{q^{\frac{1}{2}}}\right)^4 = 0.$$

Let $$q^{\frac{1}{2}} - q^{-\frac{1}{2}} = \frac{4}{(\theta - 1)^{\frac{1}{2}}},$$

then $$\theta^3 - M(\theta - 1) = 0,$$

which determines θ. And then

$$q = \frac{7 + \theta + 4\,(3 + \theta)^{\frac{1}{2}}}{\theta - 1}.$$

Suppose $q = a$ is one of the values of q; the equation becomes

$$\frac{(q^2 + 14q + 1)^3}{q.(q - 1)^4} = \frac{(a^2 + 14a + 1)^3}{a\,(a - 1)^4}$$

$$= \frac{(\beta^8 + 14\beta^4 + 1)^3}{\beta^4.(\beta^4 - 1)^4}, \quad \text{if } a = \beta^4.$$

Now if $q = \left(\frac{1-\beta}{1+\beta}\right)^4$,

$$(q^2 + 14q + 1) = \frac{16\,(\beta^8 + 14\beta^4 + 1)}{(1+\beta)^8}, \quad q - 1 = -\frac{8\beta\,(1+\beta^2)}{(1+\beta)^4},$$

which satisfy the above equation: hence also, identically,

$$(q^2 + 14q + 1)^3 - q\,(q-1)^4 . \frac{(\beta^8 + 14\beta^4 + 1)^3}{\beta^4 . (\beta^4 - 1)^4}$$

$$= (q - \beta^4)\left(q - \frac{1}{\beta^4}\right)\left\{q - \left(\frac{1-\beta}{1+\beta}\right)^4\right\}\left\{q - \left(\frac{1+\beta}{1-\beta}\right)^4\right\}$$

$$\left\{q - \left(\frac{1-\beta i}{1+\beta i}\right)^4\right\}\left\{q - \left(\frac{1+\beta i}{1-\beta i}\right)^4\right\};$$

or the values of q take the form

$$\beta^4,\ \frac{1}{\beta^4},\ \left(\frac{1-\beta}{1+\beta}\right)^4,\ \left(\frac{1+\beta}{1-\beta}\right)^4,\ \left(\frac{1-\beta i}{1+\beta i}\right)^4,\ \left(\frac{1+\beta i}{1-\beta i}\right)^4.$$

(Comp. *Abel. Œuv.* tom. I. p. 310).

The equation $\qquad \theta^3 - M\theta + M = 0$

has its three roots real if $27 - 4M$ is negative, and only a single real root if $27 - 4M$ is positive. Writing the equation under the form

$$(\theta + 3)^3 - 9\,(\theta + 3)^2 + (27 - M)\,(\theta + 3) - (27 - 4M) = 0,$$

we see that in the former case θ has two values greater than -3, and a single value less than -3. Writing the equation under the form

$$(\theta - 1)^3 + 3\,(\theta - 1)^2 + (3 - M)(\theta - 1) + 1 = 0, \quad (3{-}M \text{ is negative})$$

the positive roots are both greater than 1. Hence, in this case, q has four positive values and two imaginary ones. In the second case θ has a single real value, which is greater than -3 and less than 1. Hence q has two negative values and four imaginary ones. In the former case, $I^3 - 27J^2$ is positive, and the function U has either four imaginary factors or four real ones. In the second case, $I^3 - 27J^2$ is negative, or the function U has two real and two imaginary factors.

[redacted]

NOTE ON THE MAXIMA AND MINIMA OF FUNCTIONS OF THREE VARIABLES.

By Arthur Cayley, M.A., Fellow of Trinity College.

If A, B, C, F, G, H, be any real quantities, such that

$$BC + CA + AB - F^2 - G^2 - H^2,$$

and $(A + B + C)(ABC - AF^2 - BG^2 - CH^2 + 2FGH)$

are positive; the six quantities

$$BC - F^2,\ CA - G^2,\ AB - H^2,\ AK,\ BK,\ CK,$$

(where $K = ABC - AF^2 - BG^2 - CH^2 + 2FGH$)

are all of them positive. It is unnecessary to point out the connection of this property with the theory of maxima and minima.

To demonstrate this, writing as usual

$$\begin{aligned} BC - F^2 &= A', & GH - AF &= F', \\ CA - G^2 &= B', & HF - BG &= G', \\ AB - H^2 &= C', & FG - CH &= H', \end{aligned}$$

and K as above: then if A'', B'', C'', F'', G'', H'', K' be formed from A', B', C', F', G', H', as these and K are from A, B, C, F, G, H, we have the well known formulæ

$$\begin{aligned} A'' &= KA, & F'' &= KF, & K' &= K^2. \\ B'' &= KB, & G'' &= KG, & & \\ C'' &= KC, & H'' &= KH, & & \end{aligned}$$

It is required to show that if $A' + B' + C'$ and $A'' + B'' + C''$ are positive, A', B', C', A'', B'', C'' are so likewise.

Consider the cubic equation

$$(A'-k)(B'-k)(C'-k)-(A'-k)F'^2-(B'-k)G'^2-(C'-k)H'^2 + 2F'G'H' = 0,$$

the roots of which are all real. By the formulæ just given this may be written

$$k^3 - k^2(A' + B' + C') + k\,.\,(A'' + B'' + C'') - K^2 = 0;$$

and the terms of this equation are alternately positive and negative; *i.e.* the roots are all positive. Hence the roots of the limiting equation

$$(B' - k)(C' - k) - F'^2 = 0$$

are positive, *i.e.* $B' + C'$ and $B'C'$ are positive: but from the second condition B', C' are of the same sign. Consequently of the same sign with $B' + C'$ or positive. Also $A'' = B'C' - F'^2$

repulsion exercised by a charged conductor on a point near its surface.*

The memoirs of Poisson, on the mathematical theory, contain the analytical determination of the distribution of electricity on two conducting spheres placed near one another, the solution being worked out in numbers in the case of two equal spheres in contact, which had been investigated experimentally by Coulomb (as well as in another case, not examined by Coulomb, which is given as a specimen of the numerical results that may be deduced from the formulæ). The calculated ratios of the intensities at different points of the surface he is therefore enabled to compare with Coulomb's measurements, and he finds an agreement which is quite as close as could be expected, when we consider the excessively difficult and precarious nature of quantitative experiments in electricity: but the most remarkable confirmation of the theory from these researches is the entire agreement of the principal features, even in some very singular phenomena, of the experimental results with the theoretical deductions. For a complete account of the experiments we must refer to Coulomb's fifth memoir (*Histoire de l' Académie*, 1787), and for the mathematical investigations to the first and second memoirs of Poisson (*Mémoires de l' Institut*, 1811), or to the treatise on Electricity in the *Encyclopædia Metropolitana*, where the substance of Poisson's first memoir is given.

The mathematical theory received by far the most complete development which it has hitherto obtained, in Green's *Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*,† in which a series of general theorems were demonstrated, and many interesting applications made to actual problems.‡

Of late years some distinguished experimentalists have begun to doubt the truth of the laws established by Coulomb, and have made extensive researches with a view to discover the laws of certain phenomena which they considered incompatible with his theory. The most remarkable works of this kind have been undertaken independently by Mr. Snow Harris and Mr. Faraday, and in their memoirs, published in the *Philosophical Transactions*, we find detailed accounts of their researches. All the experiments, however, which they have made, having direct reference to the distribution of electricity in equilibrium, are, I think, in full accordance with the laws of Coulomb, and must therefore, instead of objections to his theory, be considered as confirming it. As however many

* See Note II. † Nottingham, 1828. ‡ See Note III.

[redacted]

Mr. Harris's experiments, it will be so slight on the unopposed portions that it could not be perceived without experiments of a very refined nature, such as might be made by the proof plane of Coulomb, which is in fact, with a slight modification, the instrument employed by Mr. Faraday in the investigation. Now to the degree of approximation to which the intensity on the unopposed parts may be neglected, the laws observed by Mr. Harris when the opposed surfaces are plane may be readily deduced from the mathematical theory. Thus let v be the potential in the interior of the charged body, A, a quantity which will depend solely on the state of the interior coating of the battery with which in Mr. Harris's experiments A is connected, and will therefore be sensibly constant for different positions of A relative to the uninsulated opposed body, B. Let a be the distance between the plane opposed faces of A and B, and let S be the area of the opposed parts of these faces, which will in general be the area of the smaller, if they be unequal. When the distance a is so small that we may entirely neglect the intensity on all the unopposed parts of the bodies, it is readily shewn from the mathematical theory that (since the difference of the potentials at the surfaces of A and B is v) the intensity of the electricity produced by induction at any point of the portion of the surface of B which is opposed to A, is $\frac{v}{4\pi a}$, the intensity at any point which is not so situated being insensible. Hence the attraction on any small element ω, of the portion S of the surface of B, will be in a direction perpendicular to the plane and equal to $2\pi\left(\frac{v}{4\pi a}\right)^2$.* Hence the whole attraction on B is

$$\frac{v^2 S}{8\pi a^2}.$$

This formula expresses all the laws stated by Mr. Harris as results of his experiments in the case when the opposed surfaces are plane.

3. When the opposed surfaces are curved, for instance when A and B are equal spheres, we can make no approximation analogous to that which has led us to so simple an expression in the case of opposed planes; and we find accordingly that no such simple law for the attraction in this case has been announced by Mr. Harris. He has however found

* See *Mathematical Journal*, vol. III. p. 275.

of the bodies operated upon, being partly metal in connection with the insulated system with which the body A communicates, and partly uninsulated metal, in the fixed parts of the electrometer, and in the moveable parts by which B is supported. The general effect produced by the presence of such bodies in disturbing the observed law of force, must be to make it diminish less rapidly with the distance when A and B are separated by a considerable interval: and it is probably owing, at least in part, to such disturbing causes that Mr. Harris's results nearly agree, as far as they go, with a formula which would ultimately give for the law of force the inverse square of the distance between A and B, instead of the inverse cube.

4. The determination by the mathematical theory of the attraction or repulsion between two electrified conducting spheres has not hitherto, so far as I am aware, been attempted, and would present considerable difficulty by means of the formulæ ordinarily given for such problems. It may, however, very readily be effected by means of a general theorem on the attraction between electrified conductors, which will be given in a subsequent paper. Thus, if $F(c)$ be the force of attraction, corresponding to the distance c between the centres, in the particular case when the two spheres are equal (the radius of each being unity), and the potential in the interior of one of them is nothing (as will be the case when the body is uninsulated), the potential in the interior of the other being v, I have found the following formulæ which express $F(c)$ by a converging series.

$$\text{(A)} \qquad F(c) = v^2 c \left(\frac{P_1}{Q_1^2} + \frac{P_2}{Q_2^2} + \frac{P_3}{Q_3^2} + \text{\&c.} \right), \quad \text{where}$$

$$\text{(B)}, \qquad \begin{cases} Q_1 = c^2 - 1, \\ Q_3 = (c^2 - 2)\, Q_1 - 1, \\ Q_{n+2} = (c^2 - 2)\, Q_{n+1} - Q_n. \end{cases}$$

$$\text{(C)}, \qquad \begin{cases} P_1 = 1, \\ P_2 = 2c^2 - 3. \\ P_{n+2} = (c^2 - 2)\, P_{n+1} + (Q_{n+1} - P_n). \end{cases}$$

These formulæ enable us to calculate Q_1, Q_2, Q_3, Q_4, &c., and then P_1, P_2, P_3, P_4, &c., successively, by a simple and uniform arithmetical process, for any particular value of c. I have thus calculated the values of $\frac{F(c)}{v^2}$ in five cases, the

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cessary. Thus all conducting bodies except those operated upon, must be placed beyond the reach of influence, and the distance between the repelling bodies must be considerable with reference to their linear dimensions, so that the distribution of electricity on each may be uninfluenced by the presence of the other. Also the bodies should be spheres, so that the attraction may be the same as if the whole electricity of each were collected at its centre; and the distance to be measured will then be the distance between the centres. These conditions have been expressly mentioned by Coulomb, and they have been fulfilled, as far as possible, in his researches, as we see by the descriptions of the experiments made, which we find in his memoirs. He has thus arrived by direct measurement at the law, which we know by a mathematical demonstration,* founded upon independent experiments, to be the rigorous law of nature, for electrical action. None of these precautions however have been taken in the experiments described in Mr. Harris's memoir, and the results are accordingly unavailable for the accurate *quantitative* verification of any law, on account of the numerous unknown disturbing circumstances by which they are affected. The phenomena which he observes, however, afford *qualitative* illustrations of the mathematical theory of a very interesting nature, as may be seen from the following examples of his results.

(*a*) When the distance between the bodies is great with reference to their linear dimensions, the repulsion is inversely as the square of the distance, and directly as the product of the masses.

(*b*) When the distance is small, the action becomes apparently irregular. Thus if the quantities of electricity on the two bodies be equal, the force, which is always of repulsion, does not increase so rapidly when the bodies approach, as if it followed the law of the inverse square of the distance.

(*c*) If the charges be unequal; the repulsion ceases at a certain distance, and at all smaller distances there is attraction between the bodies.

These results are, with all their peculiarities, in full accordance with the theory of Coulomb, which indicates that, if the quantities of electricity be equal, and the bodies equal and similar, there will be repulsion in every position: but if there be any difference, however small, between the charges, the repulsion will necessarily cease, and attraction commence, before contact takes place, when one body is made to approach the other. Unless, however, the difference of the charges

* See Murphy's *Electricity*, p. 41, o- ' ... ', Art. 154.

problem in each case is determinate, and we may therefore employ the elementary principles of one theory, as theorems, relative to the other. Thus, in the paper in which these considerations are developed, Coulomb's fundamental theorem relative to electricity is applied to the theory of heat; and self-evident propositions in the latter theory are made the foundation of Green's theorems in electricity.* Now the laws of motion for heat which Fourier lays down in his *Théorie Analytique de la Chaleur*, are of that simple elementary kind which constitute a mathematical theory properly so called; and therefore, when we find corresponding laws to be true for the phenomena presented by electrified bodies, we may make them the foundation of the mathematical theory of electricity: and this may be done if we consider them merely as actual truths, without adopting any physical hypothesis, although the idea they naturally suggest is that of the propagation of some effect by means of the mutual action of contiguous particles; just as Coulomb, although his laws naturally suggest the idea of material particles attracting or repelling one another at a distance, most carefully avoids making this a *physical hypothesis*, and confines himself to the consideration of the mechanical effects which he observes and their necessary consequences.†

All the views which Faraday has brought forward, and illustrated or demonstrated by experiment, lead to this method of establishing the mathematical theory, and, as far as the analysis is concerned, it would, in most *general* propositions, be even more simple, if possible, than that of Coulomb. (Of course the analysis of *particular* problems would be identical in the two methods). It is thus that Faraday arrives at a knowledge of some of the most important of the general theorems, which, from their nature, seemed destined never to be perceived except as mathematical truths. Thus, in his theory, the following proposition is an elementary principle. Let any portion a of the surface of A be projected on B, by means of lines (which will be in general curved) possessing the property that the resultant electrical force at any point of each of them is in the direction of the tangent: the quantity of electricity produced by induction on this projection is equal to the quantity of the opposite kind of elec-

* It was not until some time after that paper was published, that I was able to add the direct analytical demonstrations of the theorems, which are given in the papers on "General Propositions in the Theory of Attraction," vol. III. pp. 189, 201, and which I have since found are the same as those originally given by Green.

† See Note I.

The result in the case of a gaseous dielectric is what would follow from Coulomb's theory, if we consider gases to be quite impermeable to electricity, and to be entirely unaffected by electrical influence. The phenomena observed with solid dielectrics, which agree with the circumstance observed by Nicholson, that the *dissimulating power* of a Leyden phial depends on the nature of the glass of which it is made, as well as on its thickness, have been by some attributed to a slight degree of conducting power, or of penetrability, possessed by solid insulators. This explanation, however, seems to be very insufficient; and besides, Faraday has estimated the nature of the effects of imperfect insulation, by independent experiments, and has established, in what seems to be a very satisfactory manner, the existence of a peculiar action in the interior of solid insulators when subjected to electrical influence. As far as can be gathered from the experiments which have yet been made, it seems probable that a dielectric, subjected to electrical influence, becomes excited in such a manner that every portion of it, however small, possesses *polarity* exactly analogous to the magnetic polarity induced in the substance of a piece of soft iron under the influence of a magnet. By means of a certain hypothesis regarding the nature of magnetic action,* Poisson has investigated the mathematical laws of the distribution of magnetism and of magnetic attractions and repulsions. These laws seem to represent in the most general manner the state of a body polarized by influence, and therefore, without adopting any particular mechanical hypothesis, we may make use of them to form a mathematical theory of electrical influence in dielectrics, the truth of which can only be established by a rigorous comparison of its results with experiment.

Let us therefore consider what would be the effect, according to this theory, which would be produced by the presence of a solid dielectric, C, placed in the space between A and B, the rest of which is occupied by air. The action of C, when

* Faraday adopts the corresponding hypothesis to explain the action of a solid dielectric, which he states thus:—"If the space round a charged globe were filled with a mixture of an insulating dielectric, as oil of turpentine or air, and small globular conductors, as shot, the latter being at a little distance from each other, so as to be insulated, then these in their condition and action exactly resemble what I consider to be the condition and action of the particles of the insulating dielectric itself. If the globe were charged, these little conductors would all be polar; if the globe were discharged, they would all return to their normal state, to be polarized again upon the recharging of the globe." (*Experimental Researches*, §. 1679.) The results of the mathematical analysis of such an action are given in the text. It may be added that the value of the coefficient k will differ sensibly from unity if the volume occupied by the small conducting balls bear a finite ratio to that occupied by the insulating medium.

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S and S', in the neighbourhood of the points considered, are

$$-\frac{1}{4\pi}\left(Q - \frac{Q}{k}\right) \text{ and } \frac{1}{4\pi}\left(Q' - \frac{Q'}{k}\right).$$

Hence, if U, U' be the potentials at S, S', due to A and B alone, and v the potential at any point P, it follows* that the potential at P, due to the polarity of the dielectric, is

$$-\left(1 - \frac{1}{k}\right) U + \left(1 - \frac{1}{k}\right) U',$$

or
$$-\left(1 - \frac{1}{k}\right) v + \left(1 - \frac{1}{k}\right) U',$$

or
$$-\left(1 - \frac{1}{k}\right) v + \left(1 - \frac{1}{k}\right) v, \text{ that is, } 0,$$

according as P is within S, within S' and without S, or without S'. Hence the total potential will be, according to the position of P,

$$v - \left(1 - \frac{1}{k}\right)(U - U'),$$

or
$$\frac{v}{k} + \left(1 - \frac{1}{k}\right) U',$$

or
$$v.$$

Hence the sole effect of the dielectric C, on the state of A and B, is to diminish the potential in the interior of the former by the quantity

$$\left(1 - \frac{1}{k}\right)(U - U').$$

If the whole space between A and B be occupied by the solid dielectric, the surfaces S and A will coincide, as also, S' and B, and therefore $U = V$, $U' = 0$. Hence the potential in the interior of A will be

$$\frac{V}{k},$$

or the fraction $\frac{1}{k}$ of the potential, with the same charge on A, and with a gaseous dielectric. From this it follows that, when the dielectric is solid, it would require, to produce a given potential in the interior of A, k times the charge which would be necessary to produce the same potential when the dielectric is gaseous, and therefore the body A in a given state, defined by the potential in its interior, produces on the interior surface of B, by induction, through the solid dielectric, a quantity of electricity k times as great as through a gaseous dielectric. On this account Faraday calls the property of a dielectric measured by k, its "specific inductive capacity."

* See Green's *Essay*, Art. 12; or, *Math. Journal*, vol. III. p. 75.

[redacted]

Also $$Q + Q' = q + q'.$$

Hence we deduce $$\sigma = \frac{k\rho + k'\rho'}{k + k'}.$$

In the experiment described, one of the dielectrics is air. Hence, to obtain the required formula, we may put $k' = 1$, in this equation, and then resolve for k.

Thus we find $$k = \frac{\sigma - \rho'}{\rho - \sigma}.$$

If only one of the apparatus be originally charged, according as it is the first or the second, we shall have

$$k = \frac{\sigma}{\rho - \sigma},$$

or $$k = \frac{\rho' - \sigma}{\sigma}.$$

If the substance examined (the dielectric of the first apparatus) be any gas, or air in a different state as to pressure or temperature from the air of the second apparatus, Faraday always finds the intensity after contact to be half the original intensity, and hence for every gaseous body $k = 1$.

If the dielectric of the first apparatus be solid, the intensity after contact is found to be greater than half the original intensity when the first, and less than half when the second is the apparatus originally charged. Hence for a solid dielectric, $k > 1$. For sulphur Faraday finds the value to be rather more than 2·2; for shell-lac, about 2; and for flint-glass, greater than 1·76.

The commonly received ideas of attraction and repulsion exercised at a distance, independently of any intervening medium, are quite consistent with all the phenomena of electrical action which have been here adduced. Thus we may consider the particles of air in the neighbourhood of electrified bodies to be entirely uninfluenced, and therefore to produce no effect in the resultant action on any point: but the particles of a solid non-conductor must be considered as assuming a polarized state when under the influence of free electricity, so as to exercise attractions or repulsions on points at a distance, which, with the action due to the charged surfaces, produce the resultant force at any point. It is, no doubt, possible that such forces at a distance may be discovered to be produced entirely by the action of contiguous particles of some intervening medium, and we have an analogy for this in the case of heat, where certain effects which follow the same laws are undoubtedly propagated

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ce plan, on fait la même chose que si l' on avait découpé sur la surface un élément de même épaisseur et de même étendue que lui, et qu' on l' eût enlevé pour le porter dans la balance sans qu' il perdît rien de l' électricité qui le couvre; une fois séparé de la surface, cet élément n' aurait plus dans ses différents points qu' une épaisseur électrique moitié moindre, puisque la fluide devrait se répandre pour en couvrir les deux faces. Ce principe posé, l' expérience n' exige plus que de l' habitude et de la dextérité: après avoir touché un point de la surface avec le plan d' épreuve, on l' apporte dans la balance, où il partage son électricité avec le disque de l' aiguille qui lui est égale, et l' on observe la force de torsion à une distance connue. On répète la même experience en touchant un autre point, et le rapport des forces de torsion est le rapport des repulsions électriques; on en prend la racine carrée pour avoir le rapport des épaisseurs. Ainsi le génie de Coulomb a donné en même temps aux mathématiciens la loi fondamentale suivant laquelle la matière électrique s' attire et se repousse; et aux physiciens une balance nouvelle, et des principes d' expérience au moyen desquels ils peuvent en quelque sorte sonder l' épaisseur de l' électricité sur tous les corps, et determiner les pressions qu' elle exerce sur les obstacles qui l' arrêtent."

To this explanation it should be added that, when the proof plane is still very near the body to which it has been applied, the effect of mutual influence is such as to make the intensity be insensible at every point of the disc on the side next the conductor, and at each point of the conductor which is *under* the disc. It is only when the disc is removed to a considerable distance that the electricity spreads itself symmetrically on its two faces, and that the intensity at the point of the conductor to which it was applied, recovers its original value. It was the omission of this consideration that caused Coulomb to fall into the error alluded to above.

Note III.

This memoir of Green's has been unfortunately very little known, either in this country or on the continent. Some of the principal theorems in it have been re-discovered within the last few years, and published in the following works:—

Comptes Rendues for Feb. 11th, 1839, where part of the series of theorems is announced without demonstration, by Chasles.

Gauss's memoir on "General Theorems relating to Attractive and Repulsive Forces, varying inversely as the square of the distance," in the *Resultate aus den Beobachtungen des magnetischen Vereins im Jahre* 1839, *Leipsic*, 1840. (Translations of this paper have been published in *Taylor's Scientific Memoirs* for April 1842, and in the Numbers of *Liouville's Journal* for July and August 1842.)

Mathematical Journal, vol. III., Feb. 1842, in a paper "On the Uniform Motion of Heat, &c."

Additions to the Connaissance des Tems for 1845 (published June 1842), where Chasles supplies demonstrations of the theorems which he had previously announced.

I should add that it was not till the beginning of the present year (1845) that I succeeded in meeting with Green's Essay. The allusion made to his name with reference to the word "potential" (*Mathematical Journal*, vol. III. p. 190), was taken from a memoir of Murphy's, "On Definite Integrals with Physical Applications," in the *Cambridge Transactions*, where a mistaken definition of that term, as used by Green, is given.

NOTE IV.

This theorem may be proved as follows:—

Let S be any closed surface, containing no part of the electrified bodies within it, which we may conceive to be described between A and B; let P be the component in the direction of the normal, of the resultant force at any point of the surface S, and let ds be an element of the surface at the same point. Then it may be easily proved (see vol. III. p. 204), that

$$\iint P ds = 0 \quad \text{.........................} \quad (a),$$

the integrations being extended over the entire surface. Now let S be supposed to consist of three parts; the portion α, of the surface of A; its projection β, on the interior surface of B; and the surface generated by the curved lines of projection. The value of P at each point of the latter portion of S will be nothing, since the tangent at any point of a line of projection is the direction of the force. Hence, if $[\iint P ds]$, and $(\iint P ds)$ denote the values of $\iint P ds$, for the portions α and β of S, the equation (a) becomes

$$[\iint P ds] + (\iint P ds) = 0.$$

But if ρ be the intensity of the distribution on the surface A or B, at any point, we have, by Coulomb's theorem,

$$\rho = \frac{P}{4\pi}.$$

Hence $$[\iint \rho ds] + (\iint \rho ds) = 0,$$

which is the theorem quoted in the text.

MATHEMATICAL NOTES.

Solution of an Optical Problem proposed in the Senate-House Papers of 1844.

"IF a polished plane have an indefinite number of very fine concentric circular grooves turned on its surface, and light be incident on it from a luminous point, the appearance presented to the eye of an observer will be that of a bright curve; find its equation."

The solution depends very simply on the principle that, when a ray of light is reflected at any surface, the length of the course of the ray, reckoned from any point in the incident ray to any point in the reflected, is a minimum. For, in the above case, let the polished plane be the plane of xy, and the centre of the circular grooves the origin, $x_1 y_1 z_1$ the coordinates of the luminous point, $e f g$ those of the eye, and $x\,y\,0$ those of the point of incidence on the plane corresponding to the groove whose radius is r; then the length of the path of the ray is

$$\sqrt{\{(x_1 - x)^2 + (y_1 - y)^2 + z_1^2\}} + \sqrt{\{(e - x)^2 + (f - y)^2 + g^2\}},$$

which is to be a minimum subject to the condition

$$x^2 + y^2 = r^2,$$

or
$$\frac{dy}{dx} = -\frac{x}{y}.$$

Hence the condition of minimum is

$$\frac{(x_1 - x)\,y - (y_1 - y)\,x}{\{(x_1 - x)^2 + (y_1 - y)^2 + z_1^2\}^{\frac{1}{2}}} + \frac{(e - x)\,y - (f - y)\,x}{(e - x)^2 + (f - y)^2 + g^2\}^{\frac{1}{2}}} = 0 \ldots (a),$$

or

$$\frac{x_1 y - y_1 x}{\{(x_1 - x)^2 + (y_1 - y)^2 + z_1^2\}^{\frac{1}{2}}} + \frac{ey - fx}{\{(e - x)^2 + (f - y)^2 + g^2\}^{\frac{1}{2}}} = 0 \ldots (\mathrm{A});$$

which, if x and y be taken as current coordinates, is the equation required. The radicals have been allowed to remain, because if they had been expelled by squaring, the result would have comprised the case in which the *difference* of the lengths of the incident and reflected ray is a minimum, and we should then have introduced a branch of the curve which is extraneous to the problem.

H. G.

[Equation (a) obviously expresses the condition, that straight lines drawn from any point P of the bright curve to the eye and to the luminous point, make equal angles with the tangent to the circle described from C as centre through P, and in the plane (x, y).

If light from a luminous point be incident upon a polished rod of any form, it would follow directly from the law of reflection, that a bright point will be seen on the rod in every position, such that lines drawn from it to the eye and to the luminous point, make equal angles with the tangent. From this we might immediately deduce the solution of the above problem as well as of the following.

A straight polished rod revolves rapidly in a plane about a fixed point; to find the bright curve which is seen by an eye in any position, when light is incident from a luminous point.

Taking, as in the preceding problem, $(x_1, y_1, z_1,)$ for the coordinates of the luminous point Q; $(e, f, g,)$ for those of the eye E, and $(x, y, 0,)$ for those of the image P of the luminous point seen in the rod at any instant, or, which is the same, of a point in the bright curve, we have, for the condition that QP and EP may be equally inclined to the rod OP,

$$\frac{(x_1 - x)\,x + (y_1 - y)\,y}{\{(x_1 - x)^2 + (y_1 - y)^2 + z_1^2\}^{\frac{1}{2}}} + \frac{(e - x)\,x + (f - y)\,y}{\{(e - x)^2 + (f - y)^2 + g^2\}^{\frac{1}{2}}} = 0,$$

which is therefore the equation of the bright curve.]

ON HOMOGENEOUS FUNCTIONS OF THE THIRD ORDER WITH THREE VARIABLES.

By Arthur Cayley.

The following problem corresponds to the geometrical question of determining the polar reciprocal of a plane curve of the third order: the solution of it is also important, with reference to the linear transformations of homogeneous functions of three variables of the third order; reasons for which it has appeared to me worth while to obtain the completely developed result.

Let
$$3U = ax^3 + by^3 + cz^3 + 3iy^2z + 3jz^2x + 3kx^2y + 3i_1yz^2 + 3j_1zx^2 + 3k_1xy^2 + 6lxyz \;\ldots\ldots\; (1).$$

It is required to eliminate x, y, z, λ from the equations
$$U = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (2),$$
$$\left.\begin{aligned} \frac{dU}{dx} + \lambda\xi &= 0 \\ \frac{dU}{dy} + \lambda\eta &= 0 \\ \frac{dU}{dz} + \lambda\zeta &= 0 \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots\; (3).$$

From the equations (2), (3), we obtain immediately
$$\Theta = \xi x + \eta y + \zeta z = 0 \;\ldots\ldots\ldots\ldots\; (4);$$
and thence
$$\Theta x = 0, \quad \Theta y = 0, \quad \Theta z = 0 \;\ldots\ldots\ldots\; (5):$$
so that a single equation more, such as
$$\Phi = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (6),$$
where Φ is homogeneous and of the second order in x, y, z, would, in conjunction with the equations (3) and (5), enable us to eliminate linearly the seven quantities x^2, y^2, z^2, yz, zx, xy, λ. Such an equation may be thus obtained.

Let L, M, N, R, S, T, be the second differential coefficients of U, each of them divided by two. The equations (3) may be written
$$\begin{aligned} Lx + Ty + Sz + \lambda\xi &= 0, \;\ldots\ldots\ldots\; (7), \\ Tx + My + Rz + \lambda\eta &= 0, \\ Sx + Ry + Nz + \lambda\zeta &= 0. \end{aligned}$$

And joining to these the equation (4),
$$\xi x + \eta y + \zeta z = 0,$$

we have, by the elimination of x, y, z, in so far as they explicitly appear, and λ, an equation $\Phi = 0$ of the required form. Hence we may write

$$\Phi = -\begin{vmatrix} L, & T, & S, & \xi \\ T, & M, & R, & \eta \\ S, & R, & N, & \zeta \\ \xi, & \eta, & \zeta, & \end{vmatrix} \quad \ldots\ldots\ldots\ldots (8);$$

or substituting for L, M, N, R, S, T, and expanding,

$$\Phi = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy \ldots (9);$$

where

$$\left.\begin{aligned}
A &= (k_1 j - l^2)\,\xi^2 + (ja - j_1^2)\,\eta^2 + (ak_1 - k^2)\,\zeta^2 + 2\,(j_1 k - al)\,\eta\zeta \\
&\qquad + 2\,(kl - k_1 j_1)\,\zeta\xi + 2\,(lj_1 - jk)\,\eta\xi, \\
B &= (bi_1 - i^2)\,\xi^2 + (i_1 k - l^2)\,\eta^2 + (bk - k_1^2)\,\zeta^2 + 2\,(lk_1 - ik)\,\eta\zeta \\
&\qquad + 2\,(k_1 i - bl)\,\zeta\xi + 2\,(il - i_1 k_1)\,\eta\xi, \\
C &= (ci - i_1^2)\,\xi^2 + (cj_1 - j^2)\,\eta^2 + (j_1 i - l^2)\,\zeta^2 + 2\,(jl - j_1 i_1)\,\eta\zeta \\
&\qquad + 2\,(li_1 - ij)\,\zeta\xi + 2\,(i_1 j - cl)\,\eta\xi, \\
2F &= (bc - ii_1)\,\xi^2 + (i_1 j_1 + ck - 2lj)\,\eta^2 + (ki + bj_1 - 2lk_1)\,\zeta^2 \\
&\qquad + (l^2 + k_1 j - ki_1 - j_1 i)\,\eta\zeta + (k_1 i_1 - bj)\,\zeta\xi + (ij - ck_1)\,\eta\xi, \\
2G &= (ij + ck_1 - 2li_1)\,\xi^2 + (ca - jj_1)\,\eta^2 + (j_1 k_1 + ai - 2lk)\,\zeta^2 \\
&\qquad + (jk - ai_1)\,\eta\zeta + (l^2 + i_1 k - ij_1 - k_1 j)\,\zeta\xi + (i_1 j_1 - ck)\,\eta\xi, \\
2H &= (k_1 i_1 + bj - 2li)\,\xi^2 + (jk + ai_1 - 2lj_1)\,\eta^2 + (ab - kk_1)\,\zeta^2 \\
&\qquad + (j_1 k_1 - ai)\,\eta\zeta + (ki - bj_1)\,\zeta\xi + (l^2 + j_1 i - jk_1 - i_1 k)\,\eta\xi,
\end{aligned}\right\}$$

$$\ldots\ldots (10).$$

Performing the elimination indicated, the result may be represented by

$$\mathsf{F}U = \begin{vmatrix} a, & k_1, & j, & l, & j_1, & k, & \xi \\ k, & b, & i_1, & i, & l, & k_1, & \eta \\ j_1, & i, & c, & i_1, & j, & l, & \zeta \\ 2\xi, & . & . & . & \zeta, & \eta & . \\ . & 2\eta & . & \zeta & . & \xi & . \\ . & . & 2\zeta & \eta & \xi & . & . \\ A & B & C & F & G & H & . \end{vmatrix} = 0 \quad \ldots\ldots (11).$$

Partially expanding,

$$\mathsf{F}U = A\mathrm{a} + B\mathrm{b} + C\mathrm{c} + 2F\mathrm{f} + 2G\mathrm{g} + 2H\mathrm{h} \ldots (12).$$

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The values of the coefficients a, b, c, f, g, h, may be useful on other occasions: they are as follows.

$$\left.\begin{aligned}
\mathrm{a} = {} & 0\xi^4 + 2(cj_1 - j^2)\eta^4 + 2(bk - k_1^2)\zeta^4 \\
& + 2(4jl - 3i_1j_1 - ck)\eta^3\zeta + 4(k_1i - lb)\zeta^3\xi + 0\xi^3\eta \\
& + 2(4k_1l - 3ik - j_1b)\eta\zeta^3 + 0\zeta\xi^3 + 4(i_1j - lc)\xi\eta^3 \\
& + 2(3i_1k + 3j_1i - 2jk_1 - 4l^2)\eta^2\zeta^2 + 2(bi_1 - i^2)\zeta^2\xi^2 \\
& \qquad\qquad + 2(ci - i_1^2)\xi^2\eta^2 \\
& + 2(ii_1 - bc)\xi^2\eta\zeta + 4(ck_1 + i_1l - 2ij)\xi\eta^2\zeta \\
& \qquad\qquad + 4(bj + il - 2i_1k_1)\xi\eta\zeta^2. \\
\mathrm{b} = {} & 2(ci - i_1^2)\xi^4 + 0\eta^4 + 2(ak_1 - k^2)\zeta^4 \\
& + 0\eta^3\zeta + 2(4kl - 3j_1k_1 - ai)\zeta^3\xi + 4(i_1j - lc)\xi^3\eta \\
& + 4(j_1k - al)\eta\zeta^3 + 2(4i_1l - 3ji - k_1c)\zeta\xi^3 + 0\xi\eta^3 \\
& + 2(aj - j_1^2)\eta^2\zeta^2 + 2(3j_1i + 3k_1j - 2ki_1 - 4l^2)\zeta^2\xi^2 \\
& \qquad\qquad + 2(cj_1 - j^2)\xi^2\eta^2 \\
& + 4(ck + jl - 2j_1i_1)\xi^2\eta\zeta + 2(jj_1 - ca)\xi\eta^2\zeta \\
& \qquad\qquad + 4(ai_1 + j_1l - 2jk)\xi\eta\zeta^2. \\
\mathrm{c} = {} & 2(bi_1 - i^2)\xi^4 + 2(aj - j_1^2)\eta^4 + 0\zeta^4 \\
& + 4(j_1k - al)\eta^3\zeta + 0\zeta^3\xi + 2(4il - 3k_1i_1 - bj)\xi^3\eta \\
& + 0\eta\zeta^3 + 4(k_1i - bl)\zeta\xi^3 + 2(4j_1l - 3kj - i_1a)\xi\eta^3 \\
& + 2(ak_1 - k^2)\eta^2\zeta^2 + 2(bk - k_1^2)\zeta^2\xi^2 \\
& \qquad\qquad + 2(3k_1j + 3i_1k - 2ij_1 - 4l^2)\xi^2\eta^2 \\
& + 4(bj_1 + k_1l - 2ki)\xi^2\eta\zeta + 4(ai + kl - 2k_1j_1)\xi\eta^2\zeta \\
& \qquad\qquad + 2(kk_1 - ab)\xi\eta\zeta^2. \\
\mathrm{f} = {} & (ii_1 - bc)\xi^4 + 0\eta^4 + 0\zeta^4 \\
& + 2(j_1^2 - aj)\eta^3\zeta + (ab - kk_1)\zeta^3\xi + (3ck_1 - 2i_1l - ij)\xi^3\eta \\
& + 2(k^2 - ak_1)\eta\zeta^3 + (3bj - 2il - i_1k_1)\zeta\xi^3 + (ca - jj_1)\xi\eta^3 \\
& + 4(al - j_1k)\eta^2\zeta^2 + (ki + 2k_1l - 3bj_1)\zeta^2\xi^2 \\
& \qquad\qquad + (i_1j_1 + 2lj - 3ck)\xi^2\eta^2 \\
& + (4l^2 + 2i_1k + 2ij_1 - 8jk_1)\xi^2\eta\zeta + (7kj - 6j_1l - ai_1)\xi\eta^2\zeta \\
& \qquad\qquad + (7k_1j_1 - 6kl - ai)\xi\eta\zeta^2. \\
\mathrm{g} = {} & 0\xi^4 + (jj_1 - ca)\eta^4 + 0\zeta^4 \\
& + (3ai_1 - 2j_1l - jk)\eta^3\zeta + 2(k_1^2 - bk)\zeta^3\xi + (bc - ii_1)\xi^3\eta \\
& + (ab - kk_1)\eta\zeta^3 + 2(i^2 - bi_1)\zeta\xi^3 + (3ck - 2jl - j_1i_1)\xi\eta^3 \\
& + (j_1k_1 + 2lk - 3ai)\eta^2\zeta^2 + 4(bl - k_1i)\zeta^2\xi^2 \\
& \qquad\qquad + (ij + 2i_1l - 3ck_1)\eta^2\xi^2 \\
& + (7i_1k_1 - 6il - bj)\xi^2\eta\zeta + (4l^2 + 2j_1i + 2jk_1 - 8ki_1)\xi\eta^2\zeta \\
& \qquad\qquad + (7ik - 6k_1l - bj_1)\xi\eta\zeta^2.
\end{aligned}\right\} \quad \ldots\ldots (13).$$

$$
\begin{aligned}
\mathrm{h} = {} & 0\xi^4 + 0\eta^4 + (kk_1 - ab)\,\zeta^4 \\
& + (ca - jj_1)\,\eta^3\zeta + (3bj_1 - 2k_1l - ki)\,\zeta^3\xi + 2\,(i_1^2 - ci)\,\xi^3\eta \\
& + (3ai - 2kl - k_1j_1)\,\eta\zeta^3 + (bc - ii_1)\,\zeta\xi^3 + 2\,(j^2 - cj_1)\,\xi\eta^3 \\
& + (jk + 2j_1l - 3ai_1)\,\eta^2\zeta^2 + (k_1i_1 + 2li - 3bj)\,\zeta^2\xi^2 + 4\,(cl - i_1j)\,\xi^2\eta^2 \\
& + (7ji - 6i_1l - ck_1)\,\xi^2\eta\zeta + (7j_1i_1 - 6jl - ck)\,\xi\eta^2\zeta \\
& + (4l^2 + 2k_1j + 2ki_1 - 8ij_1)\,\xi\eta\zeta^2 .
\end{aligned}
\qquad \ldots\ldots (13).
$$

Substituting these values, the result after all reductions becomes

$$0 = \mathsf{F}U = \ldots\ldots\ldots\ldots\ldots \quad (14),$$

$$
\begin{aligned}
& \xi^6\,(6bcii_1 - 4i^3c - 4i_1^3b + 3i^2i_1^2 - b^2c^2) \\
& + \eta^6\,(6cajj_1 - 4j^3a - 4j_1^3c + 3j^2j_1^2 - c^2a^2) \\
& + \zeta^6\,(6abkk_1 - 4k^3b - 4k_1^3a + 3k^2k_1^2 - a^2b^2) \\
& + \eta^5\zeta\,(5ca^2i_1 - 17jj_1ai_1 + 24aj^2l + 12j_1^3i_1 - 5cajk \\
& \qquad + 12cj_1^2k - 7j^2j_1k - 12caj_1l - 12j_1^2jl) \\
& + \zeta^5\xi\,(5ab^2j_1 - 17kk_1bj_1 + 24bk^2l + 12k_1^3j_1 - 5abki \\
& \qquad + 12ak_1^2i - 7k^2k_1i - 12abk_1l - 12k_1^2kl) \\
& + \xi^5\eta\,(5bc^2k_1 - 17ii_1ck_1 + 24ci^2l + 12i_1^3k_1 - 5bcij \\
& \qquad + 12bi_1^2j - 7i^2i_1j - 12bci_1l - 12i_1^2il) \\
& + \eta\zeta^5\,(5a^2bi - 17kk_1ai + 24ak_1^2l + 12k^3i - 5baj_1k_1 \\
& \qquad + 12bk^2j_1 - 7kk_1^2j_1 - 12bakl - 12k^2k_1l) \\
& + \zeta\xi^5\,(5b^2cj - 17ii_1bj + 24bi_1^2l + 12i^3j - 5cbk_1i_1 \\
& \qquad + 12ci^2k_1 - 7ii_1^2k_1 - 12cbil - 12i^2i_1l) \\
& + \xi\eta^5\,(5c^2ak - 17jj_1ck + 24cj_1^2l + 12j^3k - 5aci_1j_1 \\
& \qquad + 12aj^2i_1 - 7jj_1^2i_1 - 12acjl - 12j^2j_1l) \\
& + \eta^4\zeta^2\,(-5ca^2i + 15jj_1ai - 46ajl^2 - 10j_1^3i + 12cakl - 12cj_1k^2 + 4j^2k^2 \\
& \qquad + 5caj_1k_1 + 5jj_1^2k_1 - 10aj^2k_1 - 34j_1^2ki_1 + 26jj_1kl + 34aj_1i_1l \\
& \qquad - 6a^2i_1^2 + 10j_1^2l^2 + 12ajki_1) \\
& + \zeta^4\xi^2\,(-5ab^2j + 15kk_1bj - 46bkl^2 - 10k_1^3j + 12abil - 12ak_1i^2 \\
& \qquad + 4k^2i^2 + 5abk_1i_1 + 5kk_1^2i_1 - 10bk^2i_1 - 34k_1^2ij_1 + 26kk_1il \\
& \qquad + 34bk_1j_1l - 6b^2j_1^2 + 10k_1^2l^2 + 12bkij_1) \\
& + \xi^4\eta^2\,(-5bc^2k + 15ii_1ck - 46cil^2 - 10i_1^3k + 12bcjl - 12bi_1j^2 + 4i^2j^2 \\
& \qquad + 5bci_1j_1 + 5ii_1^2j_1 - 10ci^2j_1 - 34i_1^2jk_1 + 26ii_1jl + 34ci_1k_1l \\
& \qquad - 6c^2k_1^2 + 10i_1^2l^2 + 12cijk_1) \\
& + \eta^2\zeta^4\,(-5ba^2i_1 + 15kk_1ai_1 - 46ak_1l^2 - 10k^3i_1 + 12baj_1l - 12bkj_1^2 \\
& \qquad + 4k_1^2j_1^2 + 5bakj + 5k_1k^2i - 10ak_1^2j - 34k^2j_1i + 26kk_1j_1l \\
& \qquad + 34akil - 6a^2i^2 + 10k^2l^2 + 12ak_1j_1i)
\end{aligned}
$$

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multiplied by a numerical factor. If U is of the form

$$U = PQR \dots\dots\dots\dots\dots (25),$$

then

$$\nabla U = \rho PQR = \rho U \dots\dots\dots\dots (26).$$

And this equation is consequently the condition of the function U being resolvable into linear factors. The equation in question resolves itself into

$$\frac{A}{a} = \frac{B}{b} = \frac{C}{c} = \frac{I}{i} = \frac{J}{j} = \frac{K}{k} = \frac{I_1}{i_1} = \frac{J_1}{j_1} = \frac{K_1}{k_1} = \frac{\Lambda}{l} \dots (27);$$

a system which must contain three independent equations only. It would be interesting to verify this *à posteriori*.

ON LINEAR TRANSFORMATIONS.*

By Arthur Cayley.

[*Continued from Vol. IV. p.* 209.]

In continuing my researches on the present subject, I have been led to a new manner of considering the question, which, at the same time that it is much more general, has the advantage of applying directly to the only case which one can possibly hope to develope with any degree of completeness, that of functions of two variables. In fact the question may be proposed, "To find all the derivatives of any number of functions, which have the property of preserving their form unaltered after any linear transformations of the variables." By Derivative I understand a function deduced in any manner whatever from the given functions, and I give the name of Hyperdeterminant Derivative, or simply of Hyperdeterminant, to those derivatives which have the property just enunciated. These derivatives may easily be expressed explicitly, by means of the known method of the separation of symbols. We thus obtain the most general expression of a hyperdeterminant. But there remains a question to be resolved, which appears to present very great difficulties, that of determining the *independent* derivatives, and the relation between these and the remaining ones. I have only succeeded in treating a very particular case of this

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i.e. the terms on the one side are respectively equal to the terms on the other. Hence if

$$\square = F(\|\Omega\|^f, \ \|\Omega'\|^{f'}, ..)$$

i.e. $\square$ a rational and integral function, homogeneous of the order f in the quantities of the series $\|\Omega\|$, homogeneous of the order f' in the quantities of the series $\|\Omega'\|$, &c., we have immediately

$$\dot{\square} = E^f E'^{f'} .. \square .$$

Or if U be any function whatever of the variables $x, y ....$ which is transformed by the linear substitutions above into $\dot{U}$, then

$$\dot{\square}\dot{U} = E^f E'^{f'} .. \square U.$$

Or the function $\square U$

is by the above definition a hyperdeterminant derivative. The symbol $\square$ may be called "symbol of hyperdeterminant derivation," or simply "hyperdeterminant symbol."

Let $A, B..$ represent the different quantities of the series $\|\Omega\|$,—$A', B'..$ those of the series' $\|\Omega'\|$, &c... Then $\square$ may be reduced to a single term, and we may write

$$\square = A^\alpha . B^\beta .. A'^{\alpha'} . B'^{\beta'} ..$$

Also U may be supposed of the form

$$U = \Theta . \Phi ..$$

where Θ, Φ are functions of the variables of one of the sets $x, y..$ of one of the sets $x', y'..$ &c. Thus Θ is of the form

$$F(x_1, y_1 .. x_1', y_1 ..),$$

and so on. The functions Θ, $\Phi ....$ may be the same or different. It may be supposed after the differentiations that several of the sets $x, y...$ or of the sets $x', y'....$ become identical. In such cases it will always be assumed that the functions Θ,.. into which these sets of variables enter, are similar; so that they become absolutely identical, when the variables they contain are made so. Thus the general expression of a hyperdeterminant is

$$\square U = A^\alpha B^\beta .. A'^{\alpha'} B'^{\beta'} .. \Theta\Phi ..$$

in which, after the differentiations, any number of the sets of variables are made equal. For instance, if all the sets $x, y..$ and all the sets $x', y'..$ are made equal, the hyperdeterminant refers to a single function $F(x, y .. x', y'..)$. In any other case it refers not to a single function but to several.

What precedes, is the general theory: it might perhaps have been made clearer b[illegible] to a particular case:

If several of the functions become identical, and for these some of the letters f are equivalent, it is clear that the derivative $\square U$ refers to a certain number of functions V_1, V_2.. the same or different, of the variables x, y; x', y'.. and besides that this derivative is homogeneous, of the degrees θ_1, θ_1'.. with respect to the differential coefficients of the orders f_1, f_1'.. &c. of V_1, (consequently homogeneous of the order $\theta_1 + \theta_1' + ..$ with respect to these differential coefficients collectively), homogeneous and of the degrees θ_2, θ_2'.... with respect to the differential coefficients of the orders f_2, f_2'.. of V_2, (consequently of the order $\theta_2 + \theta_2'$.. with respect to these collectively), and so on. The degree with respect to all the functions is of course $\theta_1 + \theta_1' .. + \theta_2 + \theta_2' + .. = p$ suppose. In general, only a single function will be considered, and it will be assumed that $\square U$ only contains the differential coefficients of the f^{th} order. In this case, the derivative is said to be of the degree p and of the order f. The most convenient classification is by degrees, rather than by orders.

Commencing with the simplest case, that of functions of the second order (and writing V, W instead of V_1, V_2), we have

$$\square VW = \overline{12}^a VW.$$

(where ξ_1, η_1 apply to V and ξ_2, η_2 to W). This will be constantly represented in the sequel by the notation

$$\overline{12}^a VW = B_a(V, W).$$

Hence, writing $\delta_x^{\,a} V = V^{,0}$ $\delta_x^{\,a-1}\delta_y V = V^{,1}$..

we have $B_a(V, W) = V^{,0}W^{,a} - \dfrac{[a]'}{[1]'} V^{,1}W^{,a-1} + ..$

And in particular, according as a is odd or even,

$$B_a(V, V) = 0$$

$$\tfrac{1}{2}B_a(V, V) = V^{,0}V^{,a} - \frac{a}{1} V^{,1}V^{,a-1} + ..$$

continued to the term which contains $V^{,\frac{1}{2}a}V^{,\frac{1}{2}a}$, the coefficient of this last term, divided by two.

Thus, for the functions $\frac{1}{2}(ax^2 + 2bxy + cy^2)$, $\frac{1}{24}(ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4)$, &c., if a be made equal to 2, 4, &c. respectively, we have the *constant* derivatives

$$ac - b^2$$
$$ae - 4bd + 3c^2$$
$$ag - 6bf + 15ce - 10d^2$$
$$ai - 8bh + 28cg - 56df + 35e^2.$$
$$\vdots$$

which have all of them the property of remaining unaltered,

In fact this becomes

$$4(ac-b^2)(a'c'-b'^2)-(ac'+a'c-2bb')^2=0,$$

which is one of the forms under which the result of the elimination of the variables from two quadratic equations may be written. This is a result for which I am indebted to Mr. Boole.

Passing to the third degree, we may consider in particular the derivatives

$$\square\, UVW = \overline{23}^a\, \overline{31}^a\, \overline{12}^a\, UVW = C_a(U, V, W)$$

Writing for shortness

$$A_r = \frac{[a]^r}{[r]^r}, \quad \delta_x^{\,2a-r}\delta_y^{\,r}\, U = U_{,r},$$

we have the general term

$$C_a(U, V, W) = \Sigma\left\{(-)^{r+s+t} A_r A_s A_t\, U_{,a+t-s}\, V_{,a+r-t}\, W_{,a+s-r}\right\},$$

where r, s, t extend from 0 to a. By changing the suffixes r, s the following more convenient formula

$$C_a(U,V,W) = \Sigma\Sigma\left\{(-)^{\sigma+\rho}\, U_{,\rho}\, V_{,\sigma}\, W_{,3a-\rho-\sigma}\, \Sigma\left[(-)^t A_{\rho-t} A_{\sigma+t-a} A_t\right]\right\},$$

where t extends from 0 to $2a$: ρ, σ, and $3a-\rho-\sigma$ must be positive and not greater than $2a$.

In particular, according as a is odd or even,

$$C_a(U,U,U) = 0,$$

$$C_a(U,U,U) = 6\Sigma\Sigma\left\{(-)^{\rho+\sigma}\, U_{,\rho}\, U_{,\sigma}\, U_{,3a-\rho-\sigma}\, \Sigma\left[(-)^t A_{\rho-t} A_{\sigma+t-a} A_t\right]\right\},$$

in omitting such values of ρ, σ for which $\rho > \sigma$ or $\sigma > 3a-\rho-\sigma$, and dividing by two the terms in which $\rho = \sigma$ or $\sigma = 3a-\rho-\sigma$, and by six the term for which $\rho = \sigma = 3a-\rho-\sigma = a$.

In particular, for functions of the fourth or eighth orders we have the constant derivatives

$$ace - ad^2 - b^2e - c^3 + 2bcd$$

$$aei - 4ibd - 4afh + 3ag^2 + 3ic^2 + 12beh - 8chd - 8bgf - 22ceg$$
$$+ 24cf^2 + 24d^2g - 36def + 15e^3.$$

The first of which is a simple determinant. Thus we have been led to the functions $ae - 4bd + 3c^2$ and $ace - ad^2 - eb^2 - c^3 + 2bcd$, which occur in my "Note sur quelques formules, &c." (*Crelle*, tom. XXIV.), and in the forms which M. Eisenstein has given for the solutions of equations of the four first degrees.

Let U be a function of the order $4a$: the derivative C may be expressed by means of the derivatives B.

For, consider the function

$$B_{4a}[U, B_{2a}(V, W)],$$

Paying attention to the signification of B, this may be written

$$\overline{1\theta}^{4\alpha}\,\overline{23}^{2\alpha}\,UVW,$$

where the symbols ξ_θ, η_θ refer to the two systems $x_2, y_2 : x_3, y_3$. Thus it is easily seen that we may write

$$\xi_\theta = \xi_2 + \xi_3, \quad \eta_\theta = \eta_2 + \eta_3, \quad \text{or} \quad \overline{1\theta} = \overline{12} + \overline{13} = \overline{12} - \overline{31},$$

whence the function becomes

$$(\overline{12} - \overline{31})^{4\alpha}\,\overline{23}^{2\alpha}\,UVW.$$

Of which all the terms vanish except

$$\frac{[4\alpha]^{2\alpha}}{[2\alpha]^{2\alpha}}\,\overline{12}^{2\alpha}\,\overline{23}^{2\alpha}\,\overline{31}^{2\alpha}\,UVW.$$

Or putting $$K = \frac{[4\alpha]^{2\alpha}}{[2\alpha]^{2\alpha}} = \frac{2^{4\alpha}\,1.3..(4\alpha - 1)}{2.4..4\alpha},$$

we have $B_{4\alpha}[U, B_{2\alpha}(V, W)] = KC_\alpha(U, V, W).$

Or in particular

$$B_{4\alpha}[U, B_{2\alpha}(U, U)] = KC_\alpha(U, U, U).$$

Thus for example, neglecting a numerical factor,

$$(ax^2 + 2bxy + cy^2)(cx^2 + 2dxy + ey^2) - (bx^2 + 2cxy + dy^2)^2$$
$$= (ac - b^2)\,x^4 + 2\,(ad - bc)\,x^3y + (ae + 2bd - 3c^2)\,x^2y^2$$
$$+ 2\,(be - cd)\,xy^3 + (ce - d^2)\,y^4.$$

And then

$$e(ac - b^2) - 4d\,\tfrac{2}{4}(ad - bc) + 6c\,\tfrac{1}{6}(ae + 2bd - 3c^2)$$
$$- 4b\,\tfrac{2}{4}(be - cd) + a(ce - d^2)$$
$$= 3\,(ace - ad^2 - be^2 - c^3 + 2bed).$$

We have likewise the singular equation

$$B_{2\alpha}(V, W) = K\left(x^{4\alpha}\frac{d}{da_{4\alpha}} - x^{4\alpha-1}y\,\frac{d}{da_{4\alpha-1}}\,..+ y^{4\alpha}\frac{d}{da_0}\right)C_\alpha(U, V, W)$$

where $U = \dfrac{1}{[4\alpha]^{4\alpha}}\left(a_0x^{4\alpha} - \dfrac{[4\alpha]^1}{1}\,a_1x^{4\alpha-1}y\,.. + a_{4\alpha}y^{4\alpha}\right)$, &c.

If however $U = V = W$, we must write

$$B_{2\alpha}(U, U) = \tfrac{1}{3}K\left(x^{4\alpha}\frac{d}{da_{4\alpha}} - x^{4\alpha-1}y\,\frac{d}{da_{4\alpha-1}}\,..+ y^{4\alpha}\frac{d}{da_0}\right)C_\alpha(U, U, U),$$

the reason of which is easily seen. This subject will be resumed in the sequel.

[redacted]

The functions C may be transformed in the same way as the functions B have been. In fact

$$C_a(U,\ V,\ W) = \overline{12}^{a-k}\,\overline{23}^{a-k}\,\overline{31}^{a-k}\ C_k(U,\ V,\ W),$$

if in particular $k = 1$.

$$C_1(U,\ V,\ W) = \begin{vmatrix} U'^0 & U'^1 & U'^2 \\ V'^0 & V'^1 & V'^2 \\ W'^0 & W'^1 & W'^2 \end{vmatrix} \quad U'^0 \text{ for } \overset{2}{U}{}'^0 \ldots$$

But in general

$\xi_1^{\rho'}\eta_1^{\rho}\xi_2^{\sigma'}\eta_2^{\sigma}\xi_3^{\tau'}\eta_3^{\tau}C_1(U, V, W)$, where $\rho+\rho'=\sigma+\sigma'=\tau+\tau'=2a-2$,

$$= C_1(\overset{2a-2}{U'^{\rho}}\overset{2a-2}{V'^{\sigma}}\overset{2a-2}{W'^{\tau}}) = \begin{vmatrix} U'^{\rho} & U'^{\rho+1} & U'^{\rho+2} \\ V'^{\sigma} & V'^{\sigma+1} & V'^{\rho+2} \\ W'^{\tau} & W'^{\tau+1} & W'^{\rho+2} \end{vmatrix} \quad U'^{\rho} \text{ for } \overset{2a}{U}{}'^{\rho}, \text{ \&c.}$$

whence $C_a(U, V, W)$

$$= \Sigma\Sigma\ \left\{(-)^{\rho+\sigma} \begin{vmatrix} U'^{\rho} & V'^{\sigma-1} & W'^{3a-\rho-\sigma-2} \\ U'^{\rho+1} & V'^{\sigma} & W'^{3a-\rho-\sigma-1} \\ U'^{\rho+2} & V'^{\sigma+1} & W'^{3a-\rho-\sigma} \end{vmatrix} \Sigma[(-)^t A_t' A'_{\rho-t} A'_{\sigma-a+t}]\right\},$$

where $A_t' = \dfrac{[a-1]^t}{[t]^t}$; t extends from 0 to $\overline{a-1}$; ρ, $\sigma - 1$, and $3a - \rho - \sigma - 2$ may have any positive values not greater than $2a - 2$.

In particular $C_a(U,\ U,\ U)$

$$= 6\Sigma\Sigma\ \left\{(-)^{\rho+\sigma} \begin{vmatrix} U'^{\rho} & U'^{\sigma-1} & U'^{3a-\rho-\sigma-2} \\ U'^{\rho+1} & U'^{\sigma} & U'^{3a-\rho-\sigma-1} \\ U'^{\rho+2} & U'^{\sigma+1} & U'^{3a-\rho-\sigma} \end{vmatrix} \Sigma[(-)^t A'_t A'_{\rho-t} A'_{\sigma-a+t}]\right\},$$

where ρ, σ need only have such values that $\rho < \sigma - 1$, $\sigma - 1 < 3a - \rho - \sigma - 2$.

In particular the derivative $aei - \ldots + 15e^3$ may be transformed into

$$\begin{vmatrix} a, & d, & g \\ b, & e, & h \\ c, & f, & i \end{vmatrix} - 3 \begin{vmatrix} a, & e, & f \\ b, & f, & g \\ c, & g, & h \end{vmatrix} - 3 \begin{vmatrix} b, & c, & g \\ c, & d, & h \\ d, & e, & i \end{vmatrix} + 6 \begin{vmatrix} b, & d, & f \\ c, & e, & g \\ d, & f, & h \end{vmatrix} - 15 \begin{vmatrix} c, & d, & e \\ d, & e, & f \\ e, & f, & g \end{vmatrix}$$

in which form it is obviously a linear function of the determinants

$$\left\Vert \begin{matrix} a, & b, & c, & d, & e, & f, & g \\ b, & c, & d, & e, & f, & g, & h \\ c, & d, & e, & f, & g, & h, & i \end{matrix} \right\Vert$$

which is true generally.

Omitting for the present the theory of derivatives of the form

$$\square\, UVW = \overline{23}^{\alpha}\,\overline{31}^{\beta}\,\overline{12}^{\gamma}\,UVW,$$

we pass on to the derivatives of the fourth degree, considering those forms in which all the differential coefficients are of the same order. We may write

$$\square\, UVWX = (\overline{12}.\overline{34})^{\alpha}\,(\overline{13}.\overline{42})^{\beta}\,(\overline{14}.\overline{23})^{\gamma}\,UVWX$$
$$= D_{\alpha,\beta,\gamma}(U, V, W, X) = D_{\alpha,\beta,\gamma}\,;$$

or if, for shortness,

$$\overline{12}\,.\,\overline{34} = \mathfrak{A},\quad \overline{13}\,.\,\overline{42} = \mathfrak{B},\quad \overline{14}\,.\,\overline{23} = \mathfrak{C},$$

we have $\qquad D_{\alpha,\beta,\gamma} = \mathfrak{A}^{\alpha}\,\mathfrak{B}^{\beta}\,\mathfrak{C}^{\gamma}.\,UVWX.$

Suppose $U = V = W = X$, and consider the derivatives which correspond to the same value f of $\alpha + \beta + \gamma$. The question is to determine how many of these are independent, and to express the remaining ones in terms of these. Since the functions become equal after the differentiations, we are at liberty before the differentiations to interchange the symbolic number 1, 2, 3, 4 in any manner whatever. We have thus

$$D_{\alpha,\beta,\gamma} = D_{\beta,\gamma,\alpha} = D_{\gamma,\alpha,\beta} = (-)^{f} D_{\alpha,\gamma,\beta} = (-)^{f} D_{\gamma,\beta,\alpha} = (-)^{f} D_{\beta,\alpha,\gamma}.$$

But the identical equation

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} = 0:$$

multiplied this by $\mathfrak{A}^{a}\,\mathfrak{B}^{b}\,\mathfrak{C}^{c}$ and applied to each term to the product $UVWX$, gives

$$D_{a+1,b,c} + D_{a,b+1,c} + D_{a,b,c+1} = 0\,;$$

whence if $a + b + c = f - 1$, we have a set of equations between the derivatives $D_{\alpha,\beta,\gamma}$ for which $\alpha + \beta + \gamma = f$. Reducing these by the conditions first found, suppose Θf is the number of divisions of an integer f into three parts, zero admissible, but permutations of the same three parts rejected. The number of derivatives is Θf, and the number of relations between them is $\Theta(f-1)$. Hence $\Theta f - \Theta(f-1)$ of these derivatives are independent: only when f is even, one of these is $D_{f,0,0}$, *i.e.* $\overline{12}^{f}\,\overline{34}^{f}.UVWX$, *i.e.* $\overline{12}^{f}UV.\,\overline{34}^{f}WX$, or $B_f(U, V)\,B_f(X, W)$, *i.e.* $[B_f(U, U)]^2$. Rejecting this, the number of independent derivatives when f is even, is $\Theta f - \Theta(f-1) - 1$. Let $E\left(\dfrac{a}{b}\right)$ be the greatest integer contained in the fraction $\dfrac{a}{b}$; the number required may be shown to be

$$E\frac{f}{6} \text{ or } E\frac{f+3}{6},$$

[redacted]

according as f is even or odd. Giving to f the six forms

$$6g,\quad 6g+1,\quad 6g+2,\quad 6g+3,\quad 6g+4,\quad 6g+5,$$

the corresponding numbers of the independent derivatives are

$$g,\quad g,\quad g,\quad g+1,\quad g,\quad g+1.$$

Thus there is a single derivative for the orders 3, 5, 6, 7, 8, 10 .... two for the orders 9, 11, 12, 13, 14, 16....&c.

When f is even, the terms $D_{f-3.3.0}$, $D_{f-6.6.0}$ .. and when f is odd, the terms $D_{f-1.1.0}$, $D_{f-4.4.0}$, $D_{f-7.7.0}$, &c. may be taken for independent derivatives: by stopping immediately before that in which the second suffix exceeds the first, the right number of terms is always obtained. Thus, when $f=9$ the independent derivatives are $D_{8.1.0}$, $D_{5.4.0}$, and we have the system of equations

$$\begin{aligned}
&D_{900}+D_{810}+D_{801}=0, && D_{621}+D_{531}+D_{522}=0,\\
&D_{810}+D_{720}+D_{711}=0, && D_{540}+D_{450}+D_{441}=0,\\
&D_{720}+D_{630}+D_{621}=0, && D_{531}+D_{441}+D_{432}=0,\\
&D_{711}+D_{621}+D_{612}=0, && D_{522}+D_{432}+D_{423}=0,\\
&D_{630}+D_{531}+D_{540}=0, && D_{432}+D_{342}+D_{333}=0,
\end{aligned}$$

which are to be reduced by

$$D_{900}=-D_{900}=0,\qquad D_{801}=-D_{810},\ \&c.$$

It is easy to form the table

$D_{200}=B_2^2$	$D_{500}=0$	$D_{700}=0$
$D_{110}=-\frac{1}{2}B_2^2$	D_{410},	D_{610}
	$D_{320}=-D_{410}$	$D_{520}=-D_{610}$
$D_{300}=0$	$D_{311}=0$	$D_{511}=0$
D_{210}	$D_{221}=0$	$D_{430}=D_{610}$
$D_{111}=0$		$D_{421}=0$
	$D_{600}=B_6^2$	$D_{331}=0$
$D_{400}=B_4^2$	$D_{510}=-\frac{1}{2}B_6^2$	$D_{222}=0$
$D_{310}=-\frac{1}{2}B_4^2$	$D_{420}=-\frac{2}{3}D_{330}+\frac{1}{6}B_6^2$	
$D_{220}=\frac{1}{2}B_4^2$	$D_{411}=\frac{2}{3}D_{330}+\frac{1}{3}B_6^2$	
$D_{211}=0$	D_{330},	
	$D_{321}=-\frac{1}{3}D_{330}-\frac{1}{6}B_6^2$	
	$D_{222}=\frac{2}{3}D_{330}+\frac{1}{3}B_6^2$	

$$\begin{aligned}
D_{800} &= B_8^2 \\
D_{710} &= -\tfrac{1}{2} B_8^2 \\
D_{620} &= -\tfrac{2}{3} D_{530} + \tfrac{1}{6} B_8^2 \\
D_{611} &= \tfrac{2}{3} D_{530} + \tfrac{1}{3} B_8^2 \\
D_{530}&, \\
D_{521} &= -\tfrac{1}{3} D_{530} - \tfrac{1}{12} B_8^2 \\
D_{440} &= -\tfrac{16}{15} D_{530} - \tfrac{1}{30} B_8^2 \\
D_{431} &= \tfrac{1}{15} D_{530} - \tfrac{1}{30} B_8^2 \\
D_{422} &= \tfrac{4}{15} D_{530} + \tfrac{2}{15} B_8^2 \\
D_{332} &= -\tfrac{2}{15} D_{530} - \tfrac{1}{15} B_8^2
\end{aligned}$$

$$\begin{aligned}
D_{900} &= 0 \\
D_{810}&, \\
D_{720} &= -D_{810} \\
D_{711} &= 0 \\
D_{630} &= \tfrac{1}{2} D_{810} - \tfrac{1}{2} D_{540} \\
D_{621} &= \tfrac{1}{2} D_{810} + \tfrac{1}{2} D_{540} \\
D_{540}&, \\
D_{531} &= -\tfrac{1}{2} D_{810} - \tfrac{1}{2} D_{540} \\
D_{522} &= 0 \\
D_{441} &= 0 \\
D_{432} &= \tfrac{1}{2} D_{810} + \tfrac{1}{2} D_{540} \\
D_{333} &= 0.
\end{aligned}$$

Whatever be the value, all the tables except the three first commence thus, according as f is even or odd,

$$\begin{aligned}
D_{f,0,0} &= B_f^2 \\
D_{f-1,1,0} &= -\tfrac{1}{2} B_f^2 \\
D_{f-2,2,0} &= -\tfrac{2}{3} D_{f-3,3,0} + \tfrac{1}{6} B_f^2 \\
D_{f-2,1,1} &= \tfrac{2}{3} D_{f-3,3,0} + \tfrac{1}{3} B_f^2 \\
D_{f-3,3,0}& \\
&\vdots
\end{aligned}
\qquad \text{or} \qquad
\begin{aligned}
D_{f,0,0} &= 0 \\
D_{f-1,1,0}& \\
D_{f-2,2,0} &= -D_{f-1,1,0} \\
D_{f-2,1,1} &= 0 \\
&\vdots
\end{aligned}$$

but beyond this I am not acquainted with the law.

To give some formulæ for the transformation of these derivatives; we have, for example,

$$\begin{aligned}
D_{f-1,1,0} &= (\overline{12}\,.\,\overline{34})^{f-1}\,\overline{13}\,.\,\overline{42}\;UUUU \\
&= \overline{13}\,.\,\overline{42}\;B_{f-1}(U,\,U)\,B_{f-1}(U,\,U).
\end{aligned}$$

But $\overline{12}\,.\,\overline{42} = \xi_1\eta_2\eta_3\xi_4 - \xi_1\xi_2\eta_3\eta_4 - \eta_1\eta_2\xi_3\xi_4 + \eta_1\xi_2\xi_3\eta_4,$

and
$$\begin{aligned}
\xi_1\eta_2\eta_3\xi_4\,B_{f-1}(U,U)\,B_{f-1}(U,U) \\
&= B_{f-1}(\xi U,\,\eta U)\,B_{f-1}(\eta U,\,\xi U) \\
&= B_{f-1}(U^{,0}U^{,1})\,B_{f-1}(U^{,1}U^{,0}),\ \text{\&c.}
\end{aligned}$$

(where $U^{,0}$, $U^{,1}$ stand for $\overset{1}{U^{,0}}\overset{1}{U^{,1}}$), &c.; or

$$\begin{aligned}
D_{f-1,1,0} = -2\,\{&B_{f-1}(U^{,0}U^{,0})\,B_{f-1}(U^{,1}U^{,1}) \\
&- B_{f-1}(U^{,0}U^{,1})\,B_{f-1}(U^{,1}U^{,0})\},
\end{aligned}$$

which reduces itself to

$$D_{f-1,1,0} = -2\,\{B_{f-1}(U^{,0}U^{,1})\}^2,$$

$$D_{f-1,1,0} = -2\,\{B_{f-1}(U^{,0}U^{,0})\,B_{f-1}(U^{,1}U^{,1}) - [B_{f-1}(U^{,0}U^{,1})]^2\},$$

according as f is even or odd.

For example, for the orders 3, 5, 7, 9, we have

$$D_{210} = -2\left\{4(ac - b^2)(bd - c^2) - (ad - bc)^2\right\},$$

$$D_{410} = -2\left\{4(ae - 4bd + 3c^2)(bf - 4ce + 3d^2) - (af - 3be + 2cd)^2\right\}$$

$$D_{610} = -2\left\{4(ag - 6bf + 15ce - 10d^2)(bh - 6cg + 15df - 10e^2) - (ah - 5bg + 9cf - 5de)^2\right\}.$$

$$D_{810} = -2\left\{4(ai - 8bh + 28cg - 56df + 35e^2)(bj - 8ci + 28dh - 56eg + 35f^2) - (aj - 7bi + 20ch - 28dg + 14ef)^2\right\}.$$

The derivatives D will be presently calculated in a completely expanded form up to the ninth order. We have, therefore, still to find the derivatives of the sixth and eighth orders and a second derivative of the ninth order. For the sixth order, the simplest method is to make use of D_{222}, which is easily seen to be equal to

$$24\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}$$

For the two others we have the general formulæ

$$D_{f-2,2,0} = 2\left\{B_{f-2}(U'^0U'^0)\,B_{f-2}(U'^2U'^2) - 4B_{f-2}(U'^0U'^1)\,B_{f-2}(U'^2U'^1) + B_{f-2}(U'^0U'^2)\,B_{f-2}(U'^2U'^0) + 2\left[B_{f-2}(U'^1U'^1)\right]^2\right\},$$

where U'^0, U'^1, U'^2 have been written for $\overset{2}{U}{}'^0$, $\overset{2}{U}{}'^1$, $\overset{2}{U}{}'^2$; a formula which is demonstrated in precisely the same way as that for $D_{f-1,1,0}$.

$$D_{f-3,3,0} = -2\left\{B_{f-3}(U'^0U'^3)\,B_{f-3}(U'^3U'^0) - 6B_{f-3}(U'^0U'^1)\,B_{f-3}(U'^3U'^2) + 6B_{f-3}(U'^0U'^2)\,B_{f-3}(U'^3U'^1) + 9B_{f-3}(U'^1U'^1)\,B_{f-3}(U'^2U'^2) - 9B_{f-3}(U'^1U'^2)\,B_{f-3}(U'^2U'^1) - B_{f-3}(U'^0U'^3)\,B_{f-3}(U'^3U'^0)\right\},$$

(in which U'^0, &c. stand for $\overset{0}{U}{}'^0$, &c.). In particular

$$D_{620} = 2\left\{4(ag - 6bf + 15ce - 10d^2)(ci - 6dh + 15eg - 10f^2) - 4(ah - 5bg + 9cf - 5de)(bi + 5ch + 9dg - 5ef) + (ai - 6bh + 16cg - 26df + 15e^2)^2 + 8(bh - 6cg + 15df - 10e^2)^2\right\},$$

$$D_{630} = -2\{4(ag - 6bf + 15ce - 10d^2)(dj - 6ei + 15fh - 10g^2)$$
$$- 6(ah - 5bg + 9cf - 5de)(cj - 5di + 9eh - 5fg)$$
$$+ 6(ai - 6bh + 16cg - 26df + 15e^2)(bj - 6ci + 16dh$$
$$- 26eg + 15f^2) + 36(bh - 6cg + 15df - 10e^2)(ci - 6dh$$
$$+ 15eg - 10f^2) - 9(bi - 5ch + 9dg - 5ef)^2 - (aj - 6bi$$
$$+ 15ch - 19dg + 9ef)^2\}.$$

Hence we have all the elements necessary for the calculation of the following table of the independent constant derivatives of the fourth degree, up to the ninth order.

$$D_{210} = -2(6abcd - 4ac^3 - 4b^3d + 3b^2c^2 - a^2d^2),$$

$$D_{410} = -2(10aebf - 16ae^2c - 16b^2df + 12aed^2 + 12c^2bf - 48c^3e$$
$$- 48d^3b + 76bcde + 32c^2d - a^2f^2 - 4acdf - 9b^2e^2),$$

$$^{*}D_{222} = 24(aceg + 2adef + 2gdbc - agd^2 - ae^3 - gc^3 - acf^2 - geb^2$$
$$- 2bd^2f - 2bcef + bde^2 + fdc^2 + b^2f^2 + e^2c^2 - 3ecd^2$$
$$+ bde^2 + fdc^2 + d^4),$$

$$D_{610} = -2(14agbh + 234bgcf + 990cedf - 375d^2e^2 - a^2h^2$$
$$- 25b^2g^2 - 81c^2f^2 - 18ahcf + 10ahde - 50bgde - 24acg$$
$$- 24b^2fh + 60agdf + 60cebh - 40age^2 - 40d^2bh$$
$$- 360bdf^2 - 36c^2eg + 240bfe^2 + 240d^2cg - 600ce^3$$
$$- 600d^2bf),$$

$$^{\dagger}D_{620} = 2(36agci + 696bfdh + 2340cge^2 + 2876d^2f^2 + 40b^2h^2$$
$$+ 544c^2g^2 + 1025e^4 - 388bgch - 340bhe^2 - 2596cgdf$$
$$- 4180dfe^2 - 16ahbi + a^2i^2 - 52aidf + 30aie^2 - 60agdh$$
$$- 60bfci + 60aeg^2 + 60iec^2 - 360befg - 360cdeh - 40agf^2$$
$$- 40d^2ci + 240bf^3 + 240hd^3 - 420cef^2 - 420egd^2$$
$$+ 20ach^2 + 20gib^2 + 20ahef + 20debi + 180bdg^2 + 180fhc^2$$
$$- 100bgef - 100dech),$$

$$D_{810} = -2(18aibj + 536bhci + 4256cgdh + 13328defg + 4704e^2f^2$$
$$- a^2j^2 - 49b^2i^2 - 400c^2h^2 - 784d^2g^2 - 40ajch + 56ajdg$$
$$- 28ajef - 392bidg + 196bief - 560chef - 32aci^2$$
$$- 32b^2hj + 112aidh + 112cgbj - 224aieg - 224dfbj$$
$$+ 140aif^2 + 140e^2bj - 896bdh^2 - 896c^2gi + 1792bheg$$
$$+ 1792dfci - 1120bhf^2 - 1120e^2ci - 6272ceg^2 - 6272d^2fh$$
$$+ 3920cgf^2 + 3920e^2dh - 7840df^3 - 7840ge^3),$$

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* $D_{330} = \frac{3}{8} D_{222} + \frac{1}{4} B^2_6$. † $D_{530} = -\frac{3}{4} D_{620} + \frac{1}{4} B^2_8$.

Equations which determine D_{330} and D_{530}, the quantities by means of which the remaining derivatives of the sixth and eighth orders have been expressed.

[redacted]

$$* D_{621} = -4(7bhci + 22cgdh + 39defg + 30e^2f^2 - 2b^2i^2 + 25c^2h^2 - 47d^2g^2 - 2ajch + 7ajdg - 5ajef + 74bgdi - 73befi - 127chef + 2aci^2 + 2b^2hj - 7aidh - 7cgbj - 22aieg - 22dfbj + 25aif^2 + 25bje^2 - 52bdh^2 - 52c^2gi + 23bgeh + 23cfdi + 70bhf^2 + 70ce^2i + 32ceg^2 + 32d^2fh + 25cgf^2 + 25dhe^2 - 50df^3 - 50ge^3 - 45agfh - 45cedj - 45bfg^2 - 45eid^2 + 27aeh^2 + 27c^2fj - 20ag^3 - 20jd^3).$$

We may now proceed to demonstrate an important property of the derivatives of the fourth degree, analogous to the one which exists for the third degree. Let U, V, W, X be functions of any order f: then, investigating the value of the expression

$$B_{2f-2\alpha}[B_\alpha(U, V),\ B_\alpha(W, X)].$$

This reduces itself in the first place to

$$\overline{\theta\phi}^{2f-2\alpha}\ \overline{12}^\alpha\ \overline{34}^\alpha\ UVWX,$$

where ξ_θ, η_θ refer to U and V, and ξ_ϕ, η_ϕ to W and X: this comes to writing $\xi_\theta = \xi_1 + \xi_2$, $\eta_\theta = \eta_1 + \eta_2$, and $\xi_\phi = \xi_3 + \xi_4$, $\eta_\phi = \eta_3 + \eta_4$; whence

$$\overline{\theta\phi} = \overline{13} + \overline{14} + \overline{23} + \overline{24},$$

or the function in question is

$$(13 + 14 + 23 + 24)^{2f-2\alpha}\ \overline{12}^\alpha\ \overline{34}^\alpha\ UVWX.$$

But all the terms of this where the sum of the indices of ξ_1, η_1 or ξ_2, η_2 or ξ_3, η_3 or ξ_4, η_4, exceed f, vanish: whence it is only necessary to consider those of the form

$$K_r(\overline{13}.\overline{42})^r\ (\overline{14}.\overline{23})^{f-\alpha-r}\ (\overline{12}.\overline{34})^\alpha\ UVWX,$$

where K_r denotes the numerical coefficient

$$\frac{(-)^r.[2f-2\alpha]^{2f-2\alpha}}{[r]^r\,[r]^r\,[f-\alpha-r]^{f-\alpha-r}\,[f-\alpha-r]^{f-\alpha-r}},$$

or $\quad B_{2f-2\alpha}[B_\alpha(U, V),\ B_\alpha(W, X)]$

$$= \Sigma\left\{K_r D_{\alpha,r,f-\alpha-r}(U, V, W, X)\right\}.$$

In particular, if $U = V = W = X$, writing also B_α for $B_\alpha(U, U)$,

$$B_{2f-2\alpha}(B_\alpha, B_\alpha) = \Sigma(K_r\ D_{\alpha,r,f-\alpha-r}).$$

$$* D_{540} = 2D_{621} + D_{810}.$$

Equation to determinate D_{540}.

If a is odd, this becomes

$$0 = \Sigma\,(K_r\ D_{a, r, f-a-r}),$$

an equation which must be satisfied identically by the relations that exist between the quantities D. If, on the contrary, a is even, we see that there are as many independent functions of the form

$$B_{2f-2a}\,(B_a,\ B_a)$$

as there are of the form D; and that these two systems may be linearly expressed, either by means of the other. Thus, for the orders 3, 5, 7, the derivatives D are respectively equal, neglecting a numerical factor, to

$$B_6\,(U^2,\ U^2),\quad B_{10}\,(U^2,\ U^2),\quad B_{14}\,(U^2,\ U^2).$$

For the sixth order they may be linearly expressed by means of

$$B_{12}\,(U^2,\ U^2),\quad B_6^2,$$

and so on. All that remains to complete the theory of the fourth degree is to find the general solution of this system of equations, as also of the system connecting the derivatives D.

Passing on to a more general property. Let $U_1,\ U_2 \ldots U_p$ be functions of the orders $f_1, f_2 \ldots f_p$; and suppose

$$\Theta\,(U_2 \ldots U_p) = \square\, U_2 \ldots U_p$$

a function of the degree f_1 in the variables: suppose that $\Theta\,(U_2 \ldots U_p)$ contains the differential coefficients of the order r_2 for U_2, r_3 for U_3, &c., so that $f_1 = (f_2 - r_2) + \ldots (f_p - r_p)$. Consider the expression

$$B_{f_1}\,\{U_1,\ \Theta\,(U_2 \ldots U_p)\},$$

which reduces itself in the first place to

$$(\overline{12} + \overline{13} \ldots + \overline{1p})^{f_1}\ \square\ U_1 U_2 \ldots U_p,$$

then to $\quad K\,(\overline{12}^{f_2-r_2}\ \overline{13}^{f_3-r_3}\ \vdots\ \overline{1p}^{f_p-r_p}\ \square\ U_1 U_2 \ldots U_p$;

where for shortness

$$K = \frac{[f_1]^{f_1}}{[f_2 - r_2]^{f_2-r_2} \ldots [f_p - r_p]^{f_p-r_p}}\,.$$

For if one of the indices were smaller another would be greater, for instance that of $\overline{12}$: and the symbols ξ_2, η_2 in $\overline{12}^{f_2-r_2-\lambda}\ \square$ would rise to an order higher than f_2, or the term would vanish. Hence, writing

$$\square' = \overline{12}^{f_2-r_2}\ \overline{13}^{f_3-r_3} \ldots \overline{1p}^{f_p-r_p}$$

and $\qquad \Theta'\,(U_1,\ U_2 \ldots,\ U_p) = \square'\, U_1 U_2 \ldots U_p),$

we have $B_{f_1}\{U_1 . \Theta(U_2 \dots U_p)\} = K\Theta'(U_1, U_2 \dots U_p)$;
i.e. the first side is a constant derivative of $U_1, U_2 \dots U_p$.

Suppose $$U_1 = \frac{1}{[f_1]^{f_1}}(a_0 x^{f_1} + \dots),$$

$$\Theta(U_2 \dots U_p) = \frac{1}{[f_1]^{f_1}}(A_0 x^{f_1} + \dots),$$

then $$K\Theta'(U_1 \dots U_p) = a_0 A_{f_1} - \frac{f_1}{1} a_1 A_{f_1-1} + \dots;$$

i.e. $A_{f_1} = K\dfrac{d}{da_0}\Theta'(U_1 \dots U_p)$, $\dfrac{f_1}{1}A_{f_1-1} = K\dfrac{d}{da_1}\Theta'(U_1, U_2 \dots U_p)$..
or finally,

$$\Theta(U_2 \dots U_p) = \frac{K}{[f_1]^{f_1}}\left(x^{f_1}\frac{d}{da_{f_1}} - x^{f_1-1}y\frac{d}{da_{f_1-1}} + \dots\right)\Theta'(U_1 \dots U_p),$$

an equation which holds good; changing, however, the numerical factor, when several of the functions $U_1 \dots U_p$ become identical. Hence the theorem: if U be a function given by

$$U = \frac{1}{[f]^f}(a_0 x^f + a_1 x^{f-1}y + \dots),$$

and Θ be any constant derivative whatever of U, then

$$\left(x^f\frac{d}{da_f} - x^{f-1}y\frac{d}{da_{f-1}} + \dots\right)\Theta$$

is a derivative of U, and its value, neglecting a numerical factor, may be found by omitting in the symbol $\square$, which corresponds to the derivative Θ, the factors which contain any one, no matter which, of the symbolic numbers. (See Note.*)

If, for example,

$$-\tfrac{1}{2}D_{210} = \Theta = 6abcd - 4ac^3 - 4bd^3 + 3b^2c^2 - a^2d^2,$$

or $$\square = \overline{12}^2 . \overline{34}^2 . \overline{13} . \overline{42};$$

then $$\left(x^3\frac{d}{da} - x^2y\frac{d}{dc} + xy^2\frac{d}{db} - y^3\frac{d}{da}\right)\Theta$$

reduces itself, omitting a numerical factor, to

$$\overline{12}^2\,\overline{13}\,UUU = -\tfrac{1}{2}B_1\{U, B_1(U.U)\}.$$

This may be compared with some formulæ of M. Eisenstein's, (*Crelle*, XXVII.); adopting his notation, we have

* Not given with the present paper.

$$\Phi = ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$

$$F = \tfrac{1}{36} B_2(\Phi, \Phi) = (ac - b^2)\, x^2 + (ad - bc)\, xy + (bd - c^2)\, y^2,$$

$$\Phi = -\tfrac{1}{2}\left(x^3 \frac{d}{da} - x^2y \frac{d}{dc} + xy^2 \frac{d}{db} - y^3 \frac{d}{da}\right) D,$$

where D is the same as Θ. Hence to the system of formulæ which he has given, we may add the two following:

$$\Phi_1 = \tfrac{1}{3}\left(\frac{d\Phi}{dx}\frac{dF}{dy} - \frac{d\Phi}{dy}\frac{dF}{dx}\right),$$

$$\Phi_1 = -\tfrac{1}{216}\left\{\frac{d^3\Phi}{dx^3}\frac{d^2\Phi}{dx^2}\frac{d\Phi}{dy} - \frac{d^3\Phi}{dx^2\,dy}\left(2\frac{d^2\Phi}{dx\,dy}\frac{d\Phi}{dy} + \frac{d^2\Phi}{dy^2}\frac{d\Phi}{dx}\right)\right.$$
$$\left. + \frac{d^3\Phi}{dx\,dy^2}\left(2\frac{d^2\Phi}{dx\,dy}\frac{d\Phi}{dx} + \frac{d^2\Phi}{dx^2}\frac{d\Phi}{dy}\right) - \frac{d^3\Phi}{dy^3}\frac{d^2\Phi}{dx^2}\frac{d\Phi}{dx}\right\},$$

the first of which explains most simply the origin of the function Φ_1.

It will be sufficient to indicate the reductions which may be applied to derivatives of the form

$$C_{a,\beta,\gamma}(U, V, W) = \overline{23}^\alpha . \overline{31}^\beta . \overline{12}^\gamma\, UVW,$$

where U, V, W are *homogeneous* functions. In fact, if

$$\xi x + \eta y = \Xi,$$

the above becomes, neglecting a numerical factor,

$$(\Xi_1 . \overline{23})^\alpha (\Xi_2 . \overline{31})^\beta (\Xi_3 . \overline{12})^\gamma\, UVW,$$

where the symbols ξ, η are supposed not to affect the x, y which enter into the expressions Ξ. But we have identically

$$\Xi_1\overline{23} + \Xi_2\overline{31} + \Xi_3\overline{12} = 0,$$

an equation which gives rise to reductions similar to those which have been found for the derivatives $D_{a,\beta,\gamma}$, but which require to be performed with care, in order to avoid inaccuracies with respect to the numerical factors. It may, however, be at once inferred, that the number of independent derivatives $D_{a,\beta,\gamma}$ is the same with that of the independent derivatives $D_{a,\beta,\gamma}$ for the same value of a, β, γ.

From similar reasonings to those by which $B\{U, B(U,U)\}$ has been found, the following general theorem may be inferred.

"The derivative of any number of the derivatives of one or more functions, or even of any number of functions of these derivatives, is itself a derivative of the original functions."

For the complete reduction of these double derivatives, it would be sufficient, theoretically, to be able to reduce to the smallest number possible, the derivatives of any given degree whatever. This has been done for the derivatives of the third degree $C_{\alpha,\beta,\gamma}$, and for those of the fourth degree, in which all the differentiations rise to the same order $(D_{\alpha,\beta,\gamma})$: it seems, however, very difficult to extend these methods even to the next simplest cases,—extensive researches in the theory of the division of numbers would probably be necessary. Important results might be obtained by connecting the theory of hyperdeterminants with that of elimination, but I have not yet arrived at anything satisfactory upon this subject. I shall conclude with the remark, that it is very easy to find a series, or rather a series of series's of hyperdeterminants of all degrees, viz. the determinants

$$\begin{vmatrix} a, & b \\ b, & c \end{vmatrix}, \quad \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}, \quad \begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix} \text{ \&c.}$$

$$\begin{vmatrix} . & a, & b, & c \\ . & b, & c, & d \\ a, & b, & c & . \\ b, & c, & d & . \end{vmatrix}, \quad \begin{vmatrix} . & a, & b, & c, & d, & e \\ . & b, & c, & d, & e, & f \\ . & c, & d, & e, & f, & g \\ a, & b, & c, & d, & e & . \\ b, & c, & d, & e, & f & . \\ c, & d, & e, & f, & g & . \end{vmatrix} \text{ \&c.} \quad \begin{vmatrix} . & . & a, & b, & c, & d \\ . & . & b, & c, & d, & e \\ . & a, & b, & c, & d & . \\ . & b, & c, & d, & e & . \\ a, & b, & c, & d & . & . \\ b, & c, & d, & e & . & . \end{vmatrix} \text{ \&c.}$$

However, these functions are not all independent; *e.g.* the last may be linearly expressed by the square of the second and the cube of $(ae - 4bd + 3c^2)$: nor do I know the symbolical form of these hyperdeterminant determinants.

INVESTIGATION OF PROPERTIES OF THE HYPERBOLA.

By Edmond R. Turner, Caius College.

The following is a mode of treating the hyperbola, in a manner similar to that given by Mr. O'Brien in a former number of the Journal. It does not require the use of an imaginary angle or quantity, as the method does which is given in his treatise on Analytical Geometry.

It is evident that we may put the equation to the hyperbola in the form of two equations, by means of a subsidiary angle ϕ, by assuming

$$x = a \sec \phi, \quad y = b \tan \phi;$$

and if ϕ and ϕ' be the angles corresponding to P and D, the extremities of two conjugate diameters (ϕ' being a subsidiary angle, for expressing the equation to the conjugate hyperbola)

$$\phi = \phi'.$$

Since the equation to the conjugate hyperbola is

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1,$$

we must for this curve assume

$$y = b \sec \phi', \quad x = a \tan \phi'.$$

Let xy, $x'y'$ be the points P and D respectively, then we have

$$\left.\begin{aligned} x &= a \sec \phi, & y &= b \tan \phi \\ x' &= a \tan \phi', & y' &= b \sec \phi' \end{aligned}\right\} \dots\dots\dots (1);$$

but,

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 0;$$

therefore, substituting from (1),

$$\sin \phi - \sin \phi' = 0,$$

or

$$\phi = \phi'.$$

If $r\theta$, $r'\theta'$ be the polar coordinates of P and D,

$$r^2 = x^2 + y^2 = a^2 \sec^2 \phi + b^2 \tan^2 \phi,$$

$$r'^2 = x'^2 + y'^2 = a^2 \tan^2 \phi + b^2 \sec^2 \phi;$$

thereforc

$$r^2 - r'^2 = a^2 - b^2.$$

Also if A be thc area of the parallelogram completed upon CP and CD,

$$\begin{aligned} A &= rr' \sin (\theta' - \theta) \\ &= xy' - x'y \\ &= ab \sec^2 \phi - ab \tan^2 \phi \\ &= ab. \end{aligned}$$

If in the expressions $x = a \sec \phi$, $y = b \tan \phi$, we put ϕ' for ϕ, and change x and y into x' and y', therefore

$$x' = a \tan \phi' = a \tan \phi = \frac{a}{b} y,$$

$$y' = b \sec \phi' = b \sec \phi = \frac{b}{a} x.$$

[redacted]

To find the evolute to the hyperbola.
The equation to the normal, being

$$y - y_1 = -\frac{dx_1}{dy_1}(x - x_1),$$

may be written

$$y - b\tan\phi = -\frac{a}{b}\sin\phi\,(x - a\sec\phi),$$

or
$$y = -x\frac{a}{b}\sin\phi + \frac{a^2+b^2}{b}\tan\phi;$$

therefore, differentiating with regard to ϕ,

$$0 = -x\frac{a}{b}\cos\phi + \frac{a^2+b^2}{b}\sec^2\phi:$$

therefore
$$\sec^3\phi = \frac{x}{\alpha}; \quad \text{where } \alpha = \frac{a^2+b^2}{a}.$$

Changing $\sec\phi, x, y, a, b$ into $\tan\phi, y, x, b, a$, respectively,

$$\tan^3\phi = \frac{y}{\beta}, \quad \text{where } \beta = \frac{a^2+b^2}{b};$$

therefore
$$\left(\frac{x}{\alpha}\right)^{\frac{2}{3}} + \left(\frac{y}{\beta}\right)^{\frac{2}{3}} = 1.$$

It is evident that, if a circle be described on the axis major, and a tangent be drawn to it from the foot of the ordinate at any point; the radius passing through the point of contact will be inclined at the angle ϕ to the axis of abscissas.

NOTE ON THE RINGS AND BRUSHES IN THE SPECTRA PRODUCED BY BIAXAL CRYSTALS.

By William Thomson, B.A.

It has been shown in this Journal (vol. III. p. 286) that if any system of isothermal plane curves be given, the orthogonal system, which is proved to be necessarily isothermal also, may in every case be determined. Thus if $v = a$ be the equation to the first system, v being a function of x and y which satisfies the equation

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0,$$

we shall have for the equation to the orthogonal system

$$u = \int \left(\frac{dv}{dy}\, dx - \frac{dv}{dx}\, dy \right) = \beta,$$

the expression under the sign of integration being in this case a complete differential, and the equation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$$

will be satisfied; results which may be readily verified.

To take an example, let Q, Q', &c. be any number of fixed points determined by the coordinates (a, b), (a', b'), &c., and let r, r', &c. be the distances of the point $P(xy)$ from those points. We may take for the equation of an isothermal system of curves

$$v = m \log r + m' \log r' + \&c. = a \ldots\ldots (1),$$

where $r^2 = (x - a)^2 + (y - b)^2$, &c., and m, m', &c. are constants.

In this case we have

$$u = m \tan^{-1} \frac{x - a}{y - b} + m' \tan^{-1} \frac{x - a'}{y - b'} + \&c. = \beta \ldots (2)$$

for the orthogonal system, which, as may be readily verified, is also isothermal.

To take a simple case, let there be only two fixed points, Q, Q', and let $m = m' = 1$. The equation of the first system becomes

$$rr' = \epsilon^a = c \ldots\ldots\ldots\ldots\ldots (3);$$

and, if we take the origin as the point of bisection of QQ', and make this line the axis of x, the equation of the second system becomes

$$\tan^{-1} \frac{x - a}{y} + \tan^{-1} \frac{x + a}{y} = \beta \ldots\ldots\ldots (4),$$

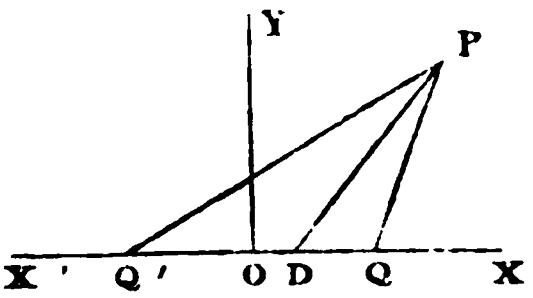
or

$$\frac{2xy}{y^2 - x^2 + a^2} = \tan \beta,$$

which may be put under the form

$$x^2 + 2kxy - y^2 = a^2 \ldots\ldots\ldots\ldots (5).$$

The equation (3) represents the series of *lemniscates* which Herschel has shown to be the forms of the rings in a biaxal crystal. Also (4) is the equation of a brush, since, if we draw PD bisecting the angle QPQ' and meeting QQ' in D, we have



$$PDQ = PQX - DPQ = PQ'X + Q'PD$$

$$= \tfrac{1}{2}(PQX + PQ'X) = \tfrac{1}{2}\left(\tan^{-1} \frac{x - a}{y} + \tan^{-1} \frac{x + a}{y} \right).$$

Hence (4) represents the locus of the points P, when the angle PDQ is constant, which is the characteristic property of a brush.

Thus we see that the rings in a biaxal crystal form a system of isothermal plane curves, and the brushes the conjugate orthogonal system.

Some curious properties of the second system, which is a series of hyperbolas, may be deduced from equation (5). Let $\frac{a}{h_1^{\frac{1}{2}}}$ and $\frac{a}{h_2^{\frac{1}{2}}}$ be the semiaxes, real and imaginary, and θ the angle which the former makes with OX. To determine h_1, h_2, and θ, we have

$$(h-1)(h+1)=k^2,$$

$$\tan^2\theta=\frac{h_1-1}{h_1+1},$$

from which we deduce $h_1=(k^2+1)^{\frac{1}{2}}$,

$$h_2=-(k^2+1)^{\frac{1}{2}},$$

$$\tan^2\theta=\frac{(k^2+1)^{\frac{1}{2}}-1}{(k^2+1)^{\frac{1}{2}}+1}.$$

Hence
$$(k^2+1)^{\frac{1}{2}}=\frac{1+\tan^2\theta}{1-\tan^2\theta}=\frac{1}{\cos 2\theta},$$

and
$$\left(\frac{a}{h_1^{\frac{1}{2}}}\right)^2=a^2\cos 2\theta.$$

Thus the second system is a series of rectangular hyperbolas whose vertices lie on the lemniscate of which the equation is

$$\rho^2=a^2\cos 2\theta.$$

By putting $y=0$ in (5), we have, for the two values of x, $\pm a$, and therefore each hyperbola passes through the points Q and Q'. The series is determined by this and the preceding property.

In addition it may be remarked that, by putting $\epsilon^2=a^2$ in (3), we find
$$(x^2+y^2)^2=2a^2(x^2-y^2),$$

or, in polar coordinates,

$$\rho^2=2a^2\cos 2\theta,$$

for the equation of one of the curves of the first system. This curve is a lemniscate similar to that which is the locus of the vertices of the second system, and similarly situated, but of different magnitude.

[redacted]

The components of the couple obtained by transferring the force on $\delta\mu$ to the origin are consequently

$$\left.\begin{array}{l}\omega^2\delta\mu\,(mz - ny)(lx + my + nz)\\ \omega^2\delta\mu\,(nx - lz)(lx + my + nz)\\ \omega^2\delta\mu\,(ly - mx)(lx + my + nz)\end{array}\right\}\ \ldots\ldots\ldots\ldots\ (2).$$

To get the components (X, Y, Z) of the resultant force at the origin, and the components (L, M, N) of the resultant couple of the centrifugal forces, we must take the sum of the preceding expressions for every element of the body. Thus, if μ be the mass of the body and $\bar{x}$, $\bar{y}$, $\bar{z}$ the coordinates of the centre of gravity: and if

$$U = \Sigma\delta\mu\, x\,(lx + my + nz),$$
$$V = \Sigma\delta\mu\, y\,(lx + my + nz),$$
$$W = \Sigma\delta\mu\, z\,(lx + my + nz),$$

we have

$$\left.\begin{array}{l}X = \omega^2\mu\,\{\bar{x} - l\,(l\bar{x} + m\bar{y} + n\bar{z})\}\\ Y = \omega^2\mu\,\{\bar{y} - m\,(l\bar{x} + m\bar{y} + n\bar{z})\}\\ Z = \omega^2\mu\,\{\bar{z} - n\,(l\bar{x} + m\bar{y} + n\bar{z})\}\end{array}\right\}\ \ldots\ldots\ (3),$$

$$\left.\begin{array}{l}L = \omega^2\,(mW - nV)\\ M = \omega^2\,(nU - lW)\\ N = \omega^2\,(lV - mU)\end{array}\right\}\ \ldots\ldots\ldots\ldots\ (4).$$

If we denote the sums

$$\Sigma\delta\mu x^2,\quad \Sigma\delta\mu y^2,\quad \Sigma\delta\mu z^2,\quad \Sigma\delta\mu yz,\quad \Sigma\delta\mu zx,\quad \Sigma\delta\mu xy$$

by $\quad F \quad G \quad H \quad A' \quad B' \quad C'$

we shall have the following expressions for U, V, W,

$$\left.\begin{array}{l}U = Fl + C'm + B'n\\ V = C'l + Gm + A'n\\ W = B'l + A'm + Hn\end{array}\right\}\ \ldots\ldots\ldots\ldots\ (5).$$

If we denote the moments of inertia of the body round the axes of coordinates by A, B, C, so that

$$A = \Sigma\delta\mu\,(y^2 + z^2) = G + H,\quad B = H + F,\quad C = F + G,$$

we have for L, M, N, the modified expressions

$$\left.\begin{array}{ll} & L = \omega^2\,(nv - mw)\\ & M = \omega^2\,(lw - nu)\\ & N = \omega^2\,(mu - lv)\\ \text{where} & u = Al - C'm - B'n\\ & v = -C'l + Bm - A'n\\ & w = -B'l - A'm + Cn\end{array}\right\}\ \ldots\ldots\ldots\ldots\ (6).$$

[redacted]

$$\left.\begin{aligned} m(B'l + A'm + Hn) - n(C'l + Gm + A'n) &= 0 \\ n(Fl + C'm + B'n) - l(B'l + A'm + Hn) &= 0 \\ l(C'l + Gm + A'n) - m(Fl + C'm + B'n) &= 0 \end{aligned}\right\} \ldots\ldots (8),$$

or

$$\left.\begin{aligned} n(-C'l + Bm - A'n) - m(-B'l - A'm + Cn) &= 0 \\ l(-B'l - A'm + Cn) - n(Al - C'm - B'n) &= 0 \\ m(Al - C'm - B'n) - l(-C'l + Bm - A'n) &= 0 \end{aligned}\right\} \ldots (9);$$

the first set being obtained from equations (4), and the second, which differ only in form, from (6). In the investigation of principal axes commonly given, the quantities F, G, H are made use of instead of the moments of inertia, A, B, C; but as by using the latter quantities a geometrical representation of the analysis, by means of Poinsot's "momental ellipsoid", may be introduced, (a different "ellipsoid of construction" being made use of in the ordinary method,) in the present article equations (9) will be employed.

If none of the quantities l, m, n vanish, (the cases in which the equations are satisfied when one or more of these quantities vanish will be examined below,) the three equations (9), on account of the peculiar form of their first members, which satisfy the relation $Ll + Mm + Nn = 0$, will be equivalent to two, which may be written thus:

$$\frac{Al - C'm - B'n}{l} = \frac{-C'l + Bm - A'n}{m} = \frac{-B'l - A'm + Cn}{n} \ldots (10).$$

Hence we infer that the diameters of the surface

$$Ax^2 + By^2 + Cz^2 - 2A'yz - 2B'zx - 2C'xy = D \ldots. (11),$$

which cut their diametral planes at right angles, are principal axes of the solid.*

7. The two equations (10) are sufficient for determining the ratios $l : m : n$; but it is convenient to assume a third unknown quantity P, to represent each member of the equations. We thus obtain the three equations

$$\left.\begin{aligned} Pl &= Al - C'm - B'n \\ Pm &= -C'l + Bm - A'n \\ Pn &= -B'l - A'm + Cn \end{aligned}\right\} \ldots\ldots\ldots (12).$$

Eliminating $l : m : n$ in the usual manner, we find

$$(A - P)(B - P)(C - P)$$
$$- A'^2(A - P) - B'^2(B - P) - C'^2(C - P) - 2A'B'C' = 0 \ldots (13),$$

* The same property may be proved of the surface

$$Fx^2 + Gy^2 + Hz^2 + 2A'yz + 2B'zx + 2C'xy = D,$$

by using equations (8) instead of (9).

and
$$\frac{l^2}{a-b} = \frac{n^2}{b-c},$$

and the locus of the centres is given by equations
$$bg + b' = 0,$$
and
$$\frac{af + a'}{(a-b)^{\frac{1}{2}}} = \frac{ch + c'}{(b-c)^{\frac{1}{2}}}.$$

The values of l and n are impossible unless b be intermediate between a and c, and therefore the planes which cut the surface in circles are perpendicular to the plane of greatest and least axes.

Cor. 2. If $a' = c' = 0$,
the cubic becomes
$$(M - b)\{(M - a)(M - c) - b'^2\} = 0.$$

I. Let $M = b$,
$$\gamma m^2 = 0, \quad m = 0,$$
$$(b - c)\,l^2 + 2b'ln + (b - a)\,n^2 = 0;$$
therefore $(b - c)(b - a) < b'^2$ if the values of $\frac{l}{n}$ be real, and different.

II.
$$(M - a)(M - c) = b'^2$$
$$\frac{l^2}{a} = -\frac{m^2}{\beta} = \frac{n^2}{\gamma},$$
hence $\frac{\beta}{a}$ must be negative, and therefore the quadratic in M must have a root between b and a, which requires that
$$(b - a)(b - c) > b'^2.$$

When this is satisfied there will be one root between b and that one of the quantities a, c which differs least from b, and the others will lie between a and c. The former root will make both $\frac{\beta}{a}$ and $\frac{\beta}{\gamma}$ negative, and the latter will make one positive and the other negative. Hence the principal sections corresponding to the former are real, and to the latter, imaginary.

Cor. 3. In the case of a surface of revolution the roots of equations (3), (4), (6) are equal; therefore
$$a\gamma = b'^2,$$
$$\beta\gamma = a'^2$$
$$a\beta = c'^2.$$

directions. If we denote now by a symbol of fractional form, such as $\frac{b}{a}$, the quotient thus obtained by dividing one line b by another line a, when directions as well as lengths are attended to, the definitional equations (26), (27), (28), (29), will take these somewhat shorter forms:*

$$\frac{c}{a} + \frac{b}{a} = \frac{c+b}{a}; \qquad \frac{c}{a} - \frac{b}{a} = \frac{c-b}{a}; \ldots\ldots (46),$$

$$\frac{c}{a} \times \frac{a}{b} = \frac{c}{b}; \qquad \frac{c}{a} \div \frac{b}{a} = \frac{c}{b}; \ \ldots\ldots\ldots (47),$$

which agree in all respects with the corresponding formulæ of ordinary algebra, and serve to fix, in the present system, the meanings of the operations +, −, ×, ÷, on what may be called *geometrical fractions*. These FRACTIONS being only other forms for what we have called *geometrical quotients* in earlier articles of this paper, we may now write the identity,

$$\frac{b}{a} = b \div a \ldots\ldots\ldots\ldots\ldots\ldots (48).$$

* On the principles alluded to in former notes, the formulæ for the addition, subtraction, multiplication, and division, of any two geometrical fractions, might be thus written:

$$\frac{D-C}{B-A} + \frac{C-A}{B-A} = \frac{D-A}{B-A},$$

$$\frac{D-A}{B-A} - \frac{C-A}{B-A} = \frac{D-C}{B-A},$$

$$\frac{D-A}{C-A} \times \frac{C-A}{B-A} = \frac{D-A}{B-A},$$

$$\frac{D-A}{B-A} \div \frac{C-A}{B-A} = \frac{D-A}{C-A};$$

A, B, C, D being symbols of any four points of space, and B − A being a symbol of the straight line drawn to B from A. If we denote this line by the biliteral symbol BA, we obtain the following somewhat shorter forms, which do not however all agree so closely with the forms of ordinary algebra:

$$\frac{DC}{BA} + \frac{CA}{BA} = \frac{DA}{BA},$$

$$\frac{DA}{BA} - \frac{CA}{BA} = \frac{DC}{BA},$$

$$\frac{DA}{CB} \times \frac{CA}{BA} = \frac{DA}{BA},$$

$$\frac{DA}{BA} \div \frac{CA}{BA} = \frac{DA}{CA}.$$

[redacted]

be apprehended, on other occasions, from this use of the letter S, and if the abridged word Scal. should be thought too long, the sign $\underset{e}{S}$ might be employed.) Eliminating the four symbols b_1, c_1, d_1, d, between the first equation (59), the first equation (60), and the three equations (61), we obtain the result

$$S\left(\frac{c}{a}+\frac{b}{a}\right)=S\frac{c}{a}+S\frac{b}{a}\ldots\ldots\ldots\ldots(62);$$

in which, by the foregoing article, $\frac{b}{a}$ and $\frac{c}{a}$ may represent any two geometrical fractions: so that we may write generally

$$S\left(\frac{h}{g}+\frac{f}{e}\right)=S\frac{h}{g}+S\frac{f}{e}\ldots\ldots\ldots\ldots(63),$$

and may enunciate in words the same result by saying, that the *scalar of the sum* of any two such fractions is equal to the *sum of the scalars*. In like manner, the three other projections b_2, c_2, d_2, being each perpendicular to a, the three other partial quotients, which enter into the second equation (60), are what we have already called *vectors* in this paper, or more fully they are the vector parts of the three quotients in the first equation (59); so that we may write

$$\frac{b_2}{a}=V\frac{b}{a},\quad \frac{c_2}{a}=V\frac{c}{a},\quad \frac{d_2}{a}=V\frac{d}{a}\ldots\ldots(64),$$

V being here used, as in a former article, for the characteristic of the operation of *taking the vector part;* we have, therefore,

$$V\left(\frac{c}{a}+\frac{b}{a}\right)=V\frac{c}{a}+V\frac{b}{a}\ldots\ldots\ldots(65),$$

$$V\left(\frac{h}{g}+\frac{f}{e}\right)=V\frac{h}{g}+V\frac{f}{e}\ldots\ldots\ldots(66),$$

and may assert that the *vector of the sum* of any two geometrical fractions is equal to the *sum of the vectors*. These formulæ (63) and (66) are important in the present system; they are however, as we see, only symbolical expressions of those very simple geometrical principles from which they have been derived, through the medium of the equations (58); namely, the principles that, *whether on a line or on a plane*, the *projection of a sum* of lines is equal to the *sum of the projections*, if the word *sum* be suitably interpreted. The analogous interpretation of a *difference* of lines, combined with similar considerations, gives in like manner the formulæ

rections, and therefore without affecting their mutual relation as summands and sum, into coincidence with three other lines b_3, c_3, d_3, such that

$$d_3 = c_3 + b_3 \ldots\ldots\ldots\ldots\ldots\ldots (69);$$

and these three new lines will be the three indices required. For a right-handed rotation through a right angle, round the line b_3 as an axis, would bring the line a into the direction originally occupied by b_2; and the length of b_2 is to the length of a in the same ratio as the length of b_3 to the assumed unit of length; therefore b_3 is, in the sense of the 7th article, the index of the vector quotient $\frac{b_2}{a}$, that is, the index of the vector part of the fraction $\frac{b}{a}$, or $\frac{f}{e}$; and similarly for the indices of the two other fractions, in the first equation (59). We may therefore write, as consequences of the construction lately assigned, and of the equations (49) and (52),

$$b_3 = \mathrm{I}\frac{f}{e}; \quad c_3 = \mathrm{I}\frac{h}{g}; \quad d_3 = \mathrm{I}\left(\frac{h}{g} + \frac{f}{e}\right) \ldots\ldots (70);$$

if we agree for the present to prefix the letter I to the symbol of a geometrical fraction, as the characteristic of the operation of *taking the index of the vector part.* Eliminating now the three symbols b_3, c_3, d_3 between the four equations (69) and (70), we obtain this general formula:

$$\mathrm{I}\left(\frac{h}{g} + \frac{f}{e}\right) = \mathrm{I}\frac{h}{g} + \mathrm{I}\frac{f}{e} \ldots\ldots\ldots\ldots (71),$$

which may be thus enunciated: the *index of the vector part of the sum* of any two geometrical fractions is equal to the *sum of the indices* of the vector parts of the summands. Combining this result with the formula (63), which expresses that the scalar of the sum is the sum of the scalars, we see that the complex *operation of adding any two geometrical fractions,* of which each is determined by its scalar and by the index of its vector part, may be in general *decomposed into two* very simple but *essentially distinct operations;* namely, *first,* the operation of adding together *two numbers,* positive or negative or null, so as to obtain a third number for their sum, according to the usual rules of elementary algebra; and *second,* the operation of adding together *two lines* in space, so as to obtain a third line, according to the geometrical rules of the composition of motions, or by drawing the diagonal of a parallelogram. In like manner the operation of *taking the difference* of two

fractions may be decomposed into the two operations of taking separately the difference of two numbers, and the difference of two lines; for we can easily prove that

$$\mathrm{I}\left(\frac{h}{g}-\frac{f}{e}\right)=\mathrm{I}\,\frac{h}{g}-\mathrm{I}\,\frac{f}{e}\ \ldots\ldots\ldots\ldots(72);$$

or, in words, that the *index* (of the vector part) *of the difference* of any two fractions is equal to the *difference of the indices.* And because it has been seen that not only for numbers but also for lines, considered among themselves, any number of summands may be in any manner grouped or transposed without altering the sum; and that the sum of a scalar and a vector is equal to the sum of the same vector and the same scalar, combined in a contrary order; it follows that the *addition* of any number of geometrical fractions is an *associative* and also a *commutative* operation: in such a manner that we may now write

$$\frac{h}{g}+\frac{f}{e}=\frac{f}{e}+\frac{h}{g};\ \frac{k}{i}+\left(\frac{h}{g}+\frac{f}{e}\right)=\left(\frac{k}{i}+\frac{h}{g}\right)+\frac{f}{e}=\frac{f}{e}+\frac{h}{g}+\frac{k}{i},\ \&c.\ \ldots\ldots\ldots(73),$$

whatever straight lines in space may be denoted by e, f, g, h, i, k, &c. We may also write, concisely,

$$\mathrm{S}\Sigma=\Sigma\mathrm{S};\quad \mathrm{V}\Sigma=\Sigma\mathrm{V};\quad \mathrm{I}\Sigma=\Sigma\mathrm{I}\ \ldots\ (74);$$

$$\mathrm{S}\Delta=\Delta\mathrm{S};\quad \mathrm{V}\Delta=\Delta\mathrm{V};\quad \mathrm{I}\Delta=\Delta\mathrm{I}\ \ldots\ldots(75);$$

using Σ, Δ as the characteristics of sum and difference, while S, V, I are still the signs of scalar, vector, index.

Separation of the Scalar and Vector Parts of the Product of any two Geometrical Fractions.

11. The definitions (46), (47) of addition and multiplication of fractions, namely

$$\frac{c}{a}+\frac{b}{a}=\frac{c+b}{a},\qquad \frac{c}{a}\times\frac{a}{b}=\frac{c}{b},$$

give obviously, for any 4 straight lines a, b, c, a′, the formula

$$\left(\frac{c}{a}+\frac{b}{a}\right)\times\frac{a}{a'}=\frac{c+b}{a'}=\left(\frac{c}{a}\times\frac{a}{a'}\right)+\left(\frac{b}{a}\times\frac{a}{a'}\right)\ \ldots\ldots\ (76);$$

and this other formula of the same kind,

$$\frac{a'}{a}\times\left(\frac{c}{a}+\frac{b}{a}\right)=\frac{a'}{\frac{a}{c+b}\times a}=\left(\frac{a'}{a}\times\frac{c}{a}\right)+\left(\frac{a'}{a}\times\frac{b}{a}\right)\ \ldots.\ (77),$$

may be proved without difficulty to be a consequence of the same definitions; the operation of multiplying a line, by the quotient of two others with which it is co-planar, being interpreted by the definition (23), so as to give, in the present notation,

$$\frac{\mathrm{e}}{\mathrm{a}} \times \mathrm{a} = \mathrm{e} \,\ldots\ldots\ldots\ldots\ldots\ldots \,(78).$$

In fact, if we assume, as we may, seven new lines, $\mathrm{d}\mathrm{b}'\mathrm{c}'\mathrm{d}'\mathrm{b}''\mathrm{c}''\mathrm{d}''$, so as to satisfy the seven conditions

$$\left.\begin{array}{l} \mathrm{c} + \mathrm{b} = \mathrm{d},\ \dfrac{\mathrm{b}}{\mathrm{a}} = \dfrac{\mathrm{a}}{\mathrm{b}'},\ \dfrac{\mathrm{c}}{\mathrm{a}} = \dfrac{\mathrm{a}}{\mathrm{c}'},\ \dfrac{\mathrm{d}}{\mathrm{a}} = \dfrac{\mathrm{a}}{\mathrm{d}'}, \\ \qquad\quad \dfrac{\mathrm{b}''}{\mathrm{a}'} = \dfrac{\mathrm{a}'}{\mathrm{b}'},\ \dfrac{\mathrm{c}''}{\mathrm{a}'} = \dfrac{\mathrm{a}'}{\mathrm{c}'},\ \dfrac{\mathrm{d}''}{\mathrm{a}'} = \dfrac{\mathrm{a}'}{\mathrm{d}'}, \end{array}\right\} \ldots\ldots (79),$$

we shall have the first member of the formula (77) equal to $\frac{\mathrm{a}'}{\mathrm{a}} \times \frac{\mathrm{a}}{\mathrm{d}'} = \frac{\mathrm{a}'}{\mathrm{d}'} =$ the second member of that formula; it will therefore be equal to $\frac{\mathrm{d}''}{\mathrm{a}'}$, and consequently will be shown to be $= \frac{\mathrm{c}''}{\mathrm{a}'} + \frac{\mathrm{b}''}{\mathrm{a}'} = \frac{\mathrm{a}'}{\mathrm{c}'} + \frac{\mathrm{a}'}{\mathrm{b}'} =$ the third member of that formula, if we can show that the conditions (79) give the relation

$$\mathrm{d}'' = \mathrm{c}'' + \mathrm{b}'' \,\ldots\ldots\ldots\ldots\ldots\ldots \,(80).$$

Now those conditions show that the line a is common to the planes of b, b′, and c, c′, and that it bisects the angle between b and b′, and also the angle between c and c′; therefore the mutual inclination of the lines b′ and c′ is equal to the mutual inclination of b and c; while the lengths of the two former lines are, by the same conditions, inversely proportional to those of the two latter. And on pursuing this geometrical reasoning, in combination with the definitional meanings of the symbolic equations (79), it appears easily that the mutual inclinations of the lines b″, c″, d″, are equal to those of b′, c′, d′, and therefore to those of b, c, d; while the lengths of b″, c″, d″ are inversely proportional to those of b′, c′, d′, and therefore directly proportional to the lengths of b, c, d: since then the line d is the symbolic sum of b and c, or the diagonal of a parallelogram described with those two lines as adjacent sides, it follows that the line d″ is similarly related to b″ and c″, or that the relation (80) holds good. The formula (77) is therefore shown to be true: and although we have not *yet* proved that the multiplication of two geometrical fractions is *always* a *distributive* operation, we see at least that either

[redacted]

$\beta\alpha$, or the product of one vector by another. For this purpose we may always conceive the index $\mathrm{I}\beta$ of the vector β to be the sum of two other indices, which shall be respectively parallel and perpendicular to the index $\mathrm{I}\alpha$ of the other vector α, as follows:

$$\mathrm{I}\beta' \parallel \mathrm{I}\alpha, \quad \mathrm{I}\beta'' \perp \mathrm{I}\alpha, \quad \mathrm{I}\beta'' + \mathrm{I}\beta' = \mathrm{I}\beta \;\ldots\ldots (83);$$

and then the vector β itself will be, by the last article, the sum of the two new vectors β' and β'', and the planes of these two new vector fractions will be respectively parallel and perpendicular to the plane of the vector fraction α; consequently, the three fractions β', β'', α will be co-linear, and we shall have, by the principle (76),

$$\beta\alpha = (\beta'' + \beta')\,\alpha = \beta''\alpha + \beta'\alpha \;\ldots\ldots\ldots (84).$$

The problem of the multiplication of *any two* vectors is thus decomposed into the two simpler problems, of multiplying first *two parallel*, and secondly *two rectangular, vectors* together. If then we merely wish to separate the scalar and the vector parts, it is sufficient to observe that if, in the general formula (47), for the multiplication of any two fractions, we suppose the factors to be parallel vectors, then the line a is perpendicular to both b and c, and is also co-planar with them, so that they are necessarily parallel to each other, and the product $\frac{\mathrm{c}}{\mathrm{b}}$ is a scalar; but if, in the same general formula, we suppose the factors to be rectangular vectors, then the three lines a, b, c are themselves mutually rectangular, and the product of the fractions is a vector. Thus, in the formula (84), the partial product $\beta'\alpha$ is a scalar, but the other partial product $\beta''\alpha$ is a vector: and we may write

$$\mathrm{S}\,.\,\beta\alpha = \beta'\alpha; \quad \mathrm{V}.\,\beta\alpha = \beta''\alpha \;\ldots\ldots\ldots\ldots (85).$$

We may therefore, more generally, under the conditions (83), decompose the formula of multiplication (82) into the two following equations:

$$\left.\begin{array}{l} \mathrm{S}\,.\,(\beta + b)\,(\alpha + a) = \beta'\alpha + ba; \\ \mathrm{V}.\,(\beta + b)\,(\alpha + a) = \beta''\alpha + \beta a + b\alpha \end{array}\right\} \ldots\ldots\ldots (86).$$

Or we may write, for abridgment,

$$c = \beta'\alpha + ba; \quad \gamma = \beta''\alpha + \beta a + b\alpha \;\ldots\ldots (87);$$

and then we shall have this other equation of multiplication,

$$\gamma + c = (\beta + b)(\alpha + a) \;\ldots\ldots\ldots\ldots (88).$$

And thus the general *separation of the scalar and vector*

[redacted]

and may therefore write

$$\bar{a} = \sqrt{(-a^2)} \ldots\ldots\ldots\ldots\ldots \text{(92)},$$

$-a^2$ being here a positive number (because a^2 is negative), and $\sqrt{(-a^2)}$ being its positive or absolute *square root*, which is an entirely *determined* (and real) *number*, when the vector a, or even when the length of its index, is determined. But although we might be led to write, in like manner, from (90), the equation

$$a = \sqrt{(-\bar{a}^2)} \ldots\ldots\ldots\ldots\ldots \text{(93)},$$

yet the same principles prove that this expression, which may denote generally any *square root of a negative number*, by a suitable choice of the positive number $\bar{a}$, is equal to a *vector* a, of which the index Ia has indeed a *determined length*, but has an entirely *undetermined direction;* the symbol in the second member of the equation (93) may therefore receive (in the present system) infinitely many different geometrical representations, or constructions, though they have all one common character: and it will be a little more consistent with the analogies of ordinary algebra to write the equation under the form

$$a = (-\bar{a}^2)^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots \text{(94)},$$

using a fractional exponent which suggests a certain degree of indeterminateness, rather than a radical sign which it is often convenient to restrict to one determined value. Thus, for example, the symbol $(-1)^{\frac{1}{2}}$, or the *square root of negative unity*, will, in the present system, denote, or be geometrically constructed by, *any vector of which the index is equal to the unit of length;* that is, any geometrical fraction of which the numerator and the denominator are lines equal to each other in length, but perpendicular to each other in direction. And we see that the geometrical principle, on which this conclusion ultimately depends, is simply this: that *two successive and similar quadrantal rotations, in any arbitrary plane, reverse the direction* of any straight line in that plane. Mr. Warren, confining himself to the consideration of lines in *one fixed plane*, has been led to attribute to his geometrical representations of the square roots of negative numbers, *one fixed direction*, or rather axis, perpendicular to that other axis on which he represents square roots of positive numbers. And other authors, both before and since the publication of Mr. Warren's work,* seem to have been in like manner

* *Treatise on the Geometrical Representation of the Square Roots of Negative Quantities*, by the Rev. John Warren, Cambridge, [illegible]. See also Dr. Peacock's Treatises on Algebra, and his Re[illegible] Association, containing references to other works.

[redacted]

Again, we are allowed to suppose, in applying the same general formula of multiplication to the same case of rectangular vectors, that the index $\mathrm{I}\alpha$ of the multiplicand $\frac{\mathrm{a}}{\mathrm{b}}$ is not only parallel to the line c, but similar (and not opposite) in direction to that line; in such a manner that the rotation round c from b to a is positive: and then the rotation round b from a to c is positive, and so is the rotation round a from c to b, and also that round $-$ a from b to c; therefore the index $\mathrm{I}\beta''$ of the multiplier is similar in direction to + b, and the index $\mathrm{I}.\beta''\alpha$ of the product is similar in direction to $-$ a; consequently *the rotation round the index of the product, from the index of the multiplier to that of the multiplicand, is positive.* And although this last result has only been proved here for the case of two rectangular vectors, yet it may easily be shown, by the principles of the 11th article, to extend to the multiplication of two general geometrical fractions. For, in the notation of that article, γ denoting the vector part of the product of any two such fractions, we have, by (87),

$$\mathrm{I}\gamma = \mathrm{I}.\beta''\alpha + a\mathrm{I}\beta + b\mathrm{I}\alpha \ldots\ldots\ldots (98);$$

$\mathrm{I}\gamma$ is therefore the symbolic sum of $\mathrm{I}.\beta''\alpha$ and of two other lines which are respectively parallel to the indices of the vector parts of the two factors, and which consequently have their sum co-planar with those indices, and therefore also co-planar, by (83), with $\mathrm{I}\beta''$ and $\mathrm{I}\alpha$; consequently $\mathrm{I}\gamma$ and $\mathrm{I}.\beta''\alpha$ both lie at the same side of the plane of $\mathrm{I}\alpha$ and $\mathrm{I}\beta''$; and therefore the rotation round $\mathrm{I}\gamma$, like that round $\mathrm{I}.\beta''\alpha$, from $\mathrm{I}\beta''$ to $\mathrm{I}\alpha$, and consequently from $\mathrm{I}\beta$ to $\mathrm{I}\alpha$, is positive. Hence also the rotation round $\mathrm{I}\beta$ from $\mathrm{I}\alpha$ to $\mathrm{I}\gamma$ is positive; that is to say, in the multiplication of two general geometrical fractions, *the rotation round the index of the vector part of the multiplier, from that of the multiplicand to that of the product, is positive;* from which may immediately be deduced a remarkable consequence, already alluded to by anticipation in the 8th article, namely—that the *multiplication of two general geometrical fractions is not a commutative operation,* or that the *order of the factors is not in general indifferent;* since the index of the vector part of the product lies at one or at the other side of the plane of the indices of the vector parts of the two factors, according as those factors are taken in one or in the other order. We have, for example, by the present article, a relation of *opposition* of signs between the products of two *rectangular* vectors, taken in two opposite

It will be remembered that the indices $\mathrm{I}(\beta + \alpha)$, $\mathrm{I}(\beta - \alpha)$, of the sum and difference of the same two vectors, are symbolically equal to two different diagonals of the same parallelogram, by former articles of this paper.

(*To be continued.*)

ERRATA IN THE PRECEDING PORTION OF THIS PAPER.

In second note, page 47, *for* ordinarily *read* ordinally.
In page 50, after equation (11), *for* theory *read* theorem.
In formula (27), page 53, *for* $+$ *read* $\div$.
In page 57, *for* co-planal *read* co-planar.

ON THE INTEGRATION OF CERTAIN DIFFERENTIAL EQUATIONS.

By the Rev. Brice Bronwin.

The equations integrated in this paper are linear, and of the second order. The mode of integration is a little different from the methods hitherto employed. But the object chiefly aimed at is to give the most simple and elegant form to the more complex of the two particular integrals. For this will often admit of very different forms.

The formula $\epsilon^{ax} f\left(\dfrac{d}{dx} + a\right) y = f\left(\dfrac{d}{dx}\right) \epsilon^{ax} y$ is well known.

Change ϵ^{x} into x, $\dfrac{d}{dx}$ into $x\dfrac{d}{dx}$, and put D for $\dfrac{d}{dx}$; it becomes

$$x^a f(xD + a)\, y = f(xD)\, x^a y \;\ldots\ldots\ldots\; (a).$$

By this formula

$$Dy = x^{-1}(xD)\, y = (xD + 1)\, x^{-1} y;$$

$$D^2 y = (xD + 1)\, x^{-1} Dy = (xD + 1)\, x^{-1}(xD + 1)\, x^{-1} y$$

$$= (xD + 1)(xD + 2)\, x^{-2} y,$$

$D^3 y = (xD + 1)(xD + 2)(xD + 3)\, x^{-3} y$, &c.; and generally

$$D^n y = (xD + 1)(xD + 2) \ldots\ldots (xD + n)\, x^{-n} y^{*} \;\ldots\ldots\; (b).$$

Put in this last $D^{-n} y$ for y, and divide by the operating factors; there results

$$x^{-n} D^{-n} y = (xD + n)^{-1}(xD + n - 1)^{-1} \ldots\ldots (xD + 1)^{-1} y.$$

Multiply this by x^n; and it becomes in virtue of (a)

$$D^{-n} y = (xD)^{-1}(xD - 1)^{-1} \ldots (xD - n + 1)^{-1} x^n y \;\ldots\ldots\; (c).$$

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* [This agrees with a formula proved in Vol. I. of the 1st Series, p. 282; also in Gregory's *Examples*, p. 31.]

Multiply (*b*) and (*c*) by x^p, and change y into $x^q y$; they become

$$\left.\begin{aligned} x^p D^n x^q y &= (xD-p+1)(xD-p+2)\dots(xD-p+n)\,x^{p+q-n}y \\ x^p D^{-n} x^q y &= (xD-p)^{-1}(xD-p-1)^{-1}\dots(xD-p-n+1)\,x^{p+q+n}y \end{aligned}\right\}\dots(d).$$

These may be convenient. We will now proceed to integrate a few equations.

Let $\dfrac{d^2y}{dx^2} + qx\dfrac{dy}{dx} + qmy = 0$, ($m$ a positive integer). . . .(1).

Multiplying by x^2, this may be written

$$x^2D^2y + qx^2(xD+m)\,y = 0;$$

by (*a*) and (*b*), or by (*d*), at once this becomes

$$xD(xD-1)\,y + (xD+m-2)\,qx^2y = 0.$$

Make $y = (xD+1)(xD+2)\dots(xD+m-1)\,x^{-m+1}u = D^{m-1}u.$

Then $\quad x^2y = (xD-1)(xD)\dots\dots(xD+m-3)\,x^{-m+3}u.$

These values of y and x^2y being put in the preceding equation, after dividing by the factors common to both the terms, it becomes

$$(xD+m-1)\,x^{-m+1}u + qx^{-m+3}u = 0\,\dots\dots\dots(2).$$

I shall recur to this equation presently. It is equivalent to $\dfrac{du}{dx} + qxu = 0$, or $u = C\epsilon^{-\frac{1}{2}qx^2}$, which gives a particular integral of (1). To find the other particular integral, make $y = \epsilon^{-\frac{1}{2}qx^2}z$; and (1) will be transformed into

$$\frac{d^2z}{dx^2} - qx\frac{dz}{dx} + qmz = 0\,\dots\dots\dots\dots\dots(3).$$

Treating this in the same manner, we have

$$xD(xD-1)\,z - (xD-m-2)\,qx^2z = 0.$$

Make $z = (xD)^{-1}(xD-1)^{-1}\dots(xD-m)^{-1}x^{m+1}u = D^{-m-1}u$:
then also $x^2z = (xD-2)^{-1}(xD-3)^{-1}\dots(xD-m-2)^{-1}x^{m+3}u.$

With these values, the common factors being expunged, the preceding becomes

$$(xD-m-1)\,x^{m+1}u - qx^{m+3}u = 0,$$

or $\dfrac{du}{dx} - qxu = 0$, $u = C_1\epsilon^{\frac{1}{2}qx^2}$; and therefore $z = C_1D^{-m-1}\epsilon^{\frac{1}{2}qx^2}$.

Hence the complete integral of (1) is

$$y = C\left(\frac{d}{dx}\right)^{m-1}\epsilon^{-\frac{1}{2}qx^2} + C_1\epsilon^{-\frac{1}{2}qx^2}\left(\frac{d}{dx}\right)^{-m-1}\epsilon^{\frac{1}{2}qx^2};$$

that of (3) is given by $z = \epsilon^{\frac{1}{2}qx^2}y$.

We now return to (2). By making the factors divided by to operate upon the second member; that would have been

$$(xD - m - 1)x^{-m-2}u - qx^{-m-2}u = (xD - m - 2)^{-1}(xD - m - 3)^{-1}\dots$$
$$(xD - 1)^{-1}0 = x^{-m-2}D^{-m}x^{-2}0 = x^{-m-2}D^{-m-1}C.$$

This reduction is made by the second of (*d*) and one integration. Now this would give at once the complete integral of (1). But it would not be so simple and elegant as the above by a great deal, and simplicity of form is often a matter of great importance.

Let $\quad x^2\dfrac{d^2y}{dx^2} - m\dfrac{dy}{dx} - ry = 0,\ r = p(p+1)\ \dots\dots\ (4).$

Multiply by x, and as before by the formulæ (*a*), (*b*), or (*d*); we find

$$x\,\{xD(xD-1) - p(p+1)\}\,y - xDmy = 0,$$

or $\qquad x(xD+p)(xD-p-1)\,y - xDmy = 0,$

or $\quad (xD+p-1)(xD-p-2)\,xy - (xD)\,my = 0\ \dots\ (5),$

where we may observe that the case of $p = 1$ is integrable immediately.

Make $\quad y = (xD+1)(xD+2)\dots(xD+p-1)\,x^{-p-1}u = D^{p-1}u,$

and $\qquad xy = (xD)(xD+1)\dots(xD+p-2)\,x^{-p-2}u.$

These values put in (5), it will become, dividing by the common factors,

$(xD - p - 2)\,x^{-p-2}u - mx^{-p-1}u = 0,$ or $x^2\dfrac{du}{dx} - (m + 2px)\,u = 0,$

and $\qquad u = Cx^{2p}\varepsilon^{-\frac{m}{x}}.$

Again make

$$y = (xD)^{-1}(xD-1)^{-1}\dots(xD-p-1)^{-1}x^{p+2}u = D^{-p-2}u,$$
$$xy = (xD-1)^{-1}\dots(xD-p-2)^{-1}x^{p+3}u.$$

With these values (5) becomes

$$(xD + p - 1)\,x^{p+3}u - mx^{p+2}u = 0,$$

or $x^2\dfrac{du}{dx} - \{m - (2p+2)\,x\}\,u = 0;$ which gives $u = C_1x^{-2p-2}\varepsilon^{-\frac{m}{x}}.$

The complete integral of (5) therefore is

$$y = C\left(\frac{d}{dx}\right)^{p-1}x^{2p}\varepsilon^{-\frac{m}{x}} + C_1\left(\frac{d}{dx}\right)^{-p-2}x^{-2p-2}\varepsilon^{-\frac{m}{x}}.$$

Here again the integral is much more simple than it would be if found at once by operating upon 0 with the expunged factors.

Let $x^2 \dfrac{d^2y}{dx^2} + (m - x)\, x \dfrac{dy}{dx} + n\,(m - n - 1)\, y = 0$, n integer...(6).

From this we derive

$$\{xD\,(xD - 1) + mxD + n\,(m - n - 1)\}\, y - x\,(xD)\, y = 0:$$

or by further reduction

$$(xD + n)\,(xD + m - n - 1)\, y - (xD - 1)\, xy = 0.$$

Make $y = (xD + n)^{-1}(xD + n - 1)^{-1} ... (xD)^{-1}\, u = x^{-n} D^{-n-1} x^{-1} u$,

$$xy = (xD + n - 1)^{-1}\,(xD + n - 2)^{-1} \dots (xD - 1)^{-1}\, xu.$$

By the substitution of these values, and dividing by the factors common to both the terms, the last equation becomes

$$(xD + m - n - 1)\, u - xu = 0, \quad \text{or } x \frac{du}{dx} + (m - n - 1 - x)\, u = 0,$$

and $u = Cx^{n-m+1}\, \varepsilon^x$. There are other forms by which this particular integral might be found, but this is the simplest.

Now make $y = x^{-m} \varepsilon^x z$, and with this value of y (6) will be transformed into

$$x^2 \frac{d^2z}{dx^2} + (x - m)\, x \frac{dz}{dx} + (m - n)\,(n + 1)\, z = 0 \dots . (7).$$

This gives

$$\{xD\,(xD - m - 1) + (m - n)\,(n + 1)\}\, z + x\,(xD)\, z = 0,$$

or $$(xD + n - m)\,(xD - n - 1)\, z + (xD - 1)\, xz = 0.$$

In the last make

$$z = (xD - n)\,(xD - n + 1) \dots (xD - 1)\, u = x^{n+1} D^n x^{n-1} u,$$

$$xz = (xD - n - 1)\,(xD - n - 2) \dots (xD - 2)\, xu.$$

These values, substituted in the preceding, change it into

$$(xD + n - m)\, u + xu = 0, \quad \text{or } x \frac{du}{dx} + (n - m + x)\, u = 0,$$

$$\text{and } u = C_1 x^{m-n} \varepsilon^{-x},\ z = C_1 x^{n+1} \left(\frac{d}{dx}\right)^n x^{m-n-1} \varepsilon^{-x};$$

therefore $y = Cx^{-n} \left(\dfrac{d}{dx}\right)^{-n-1} x^{n-m}\, \varepsilon^x + C_1 x^{n-m+1}\, \varepsilon^x \left(\dfrac{d}{dx}\right)^n x^{m-n-1}\, \varepsilon^{-x}$.

This is the complete integral of (6), and $z = x^m \varepsilon^{-x} y$ will give that of (7). We might of course have found the complete integral at once here, as before indicated; but our object is to exhibit it in the simplest form.

Let $$(1 - x^2) \frac{d^2y}{dx^2} + p\,(p + 1)\, y = 0 \dots\dots . (8).$$

Multiply by x^2, and as before we find

$$xD(xD-1)\,y - x^2\,\{xD(xD-1) - p(p+1)\}\,y = 0,$$

or $$xD(xD-1)\,y - x^2\,(xD+p)(xD-p-1)\,y = 0,$$

and $$xD(xD-1)\,y - (xD+p-2)(xD-p-3)\,x^2y = 0 \ldots (9).$$

Make $y = (xD+1)(xD+2)\ldots(xD+p-1)\,x^{-p+1}u = D^{p-1}u$,

$$x^2y = (xD-1)(xD)\ldots(xD+p-3)\,x^{-p+3}u.$$

Substitute these values in the last, and it becomes

$$(xD+p-1)\,x^{-p+1}u - (xD-p-3)\,x^{-p+3}u = 0,$$

or $$(1-x^2)\frac{du}{dx} + 2pxu = 0, \text{ and } u = C(1-x^2)^p.$$

Again make

$$y = (xD)^{-1}(xD-1)^{-1}\ldots.(xD-p-1)^{-1}x^{p+2}u = D^{-p-2}u,$$

$$x^2y = (xD-2)^{-1}(xD-3)^{-1}\ldots.(xD-p-3)^{-1}x^{p+4}u.$$

Put these values in (9), and it will give

$$(xD-p-2)\,x^{p+2}u - (xD+p-2)\,x^{p+4}u = 0,$$

or $$(1-x^2)\frac{du}{dx} - (2p+2)\,xu = 0, \text{ and } u = C_1(1-x^2)^{-p-1}.$$

Consequently $y = C\left(\dfrac{d}{dx}\right)^{p-1}(1-x^2)^p + C_1\left(\dfrac{d}{dx}\right)^{-p-2}(1-x^2)^{-p-1}$,

the complete integral of (8). And this also is much more simple than it would be if found at once. We shall give one example more.

Let $x\dfrac{d^2y}{dx^2} + qx\dfrac{dy}{dx} + qmy = 0$, m a positive integer...(10).

This gives, after multiplying by x,

$$xD(xD-1)\,y + qx\,(xD+m)\,y = 0,$$

or $$xD(xD-1)\,y + (xD+m-1)\,qxy = 0.$$

Make $y = (xD+1)(xD+2)\ldots(xD+m-1)\,x^{-m+1}u = D^{m-1}u$,

$$xy = (xD)(xD+1)\ldots(xD+m-2)\,x^{-m+2}u.$$

Substitute these expressions of y and xy in the last, and it gives

$$(xD-1)\,x^{-m+1}u + qx^{-m+2}u = 0, \quad \text{or } x\frac{du}{dx} + (qx-m)\,u = 0,$$

and $$u = Cx^m\varepsilon^{-qx}.$$

To find the other particular integral make $y = \varepsilon^{-qx}z$, and (10) will be transformed into

$$x\frac{d^2z}{dx^2} - qx\frac{dz}{dx} + qmz = 0 \;\ldots\ldots\ldots\; (11).$$

From this we deduce, first

$$xD\,(xD-1)\,z - x\,(xD-m)\,qz = 0,$$

and then $\quad xD\,(xD-1)\,z - (xD-m-1)\,qxz = 0.$

Make $z = (xD)^{-1}\,(xD-1)^{-1}\ldots(xD-m)^{-1}\,x^{m+1}\,u = D^{-m-1}\,u,$

$$xz = (xD-1)^{-1}\,(xD-2)^{-1}\ldots(xD-m-1)^{-1}\,x^{m+2}\,u.$$

Put these values in the above, and we find

$$(xD-1)\,x^{m+1}\,u - qx^{m+2}\,u = 0, \text{ or } x\,\frac{du}{dx} + (m-qx)\,u = 0,$$

and $\qquad u = C_1 x^{-m}\,\varepsilon^{qx}:$

whence $\quad y = C\left(\dfrac{d}{dx}\right)^{m-1} x^{m}\,\varepsilon^{-qx} + C_1\varepsilon^{-qx}\left(\dfrac{d}{dx}\right)^{-m-1} x^{-m}\,\varepsilon^{qx},$

the complete integral of (10). And $z = \varepsilon^{qx} y$ will give that of (11).

The method of integration employed in this paper is well adapted to the integration of a certain class of partial differential equations. I shall give an example in concluding this paper. And here let D_x, D_y stand for $\dfrac{d}{dx}$ and $\dfrac{d}{dy}$ respectively.

Let $$\frac{d^2z}{dx^2} - \frac{d^2z}{dy^2} - \frac{2}{x}\,\frac{dz}{dx} = 0 \;\ldots\ldots\ldots\ldots\; (12).$$

By (*a*) and (*b*) this may be put under the form

$$(xD_x+1)\,(xD_x+2)\,x^{-2}\,z - (yD_y+1)\,(yD_y+2)\,y^{-2}\,z$$
$$- 2\,(xD_x+2)\,x^{-2}\,z = 0,$$

or $\quad (xD_x-1)\,(xD_x+2)\,x^{-2}\,z - (yD_y+1)\,(yD_y+2)\,y^{-2}\,z = 0.$

Make $\qquad z = (xD_x-1)\,u\,;$

then $\quad x^{-2}z = (xD_x+1)\,x^{-2}u,\;\; y^{-2}z = (xD_x-1)\,y^{-2}u.$

Putting these values in the preceding, and dividing by $xD_x - 1$, it becomes

$$(xD_x+1)\,(xD_x+2)\,x^{-2}u - (yD_y+1)\,(yD_y+2)\,y^{-2}u$$
$$= (xD_x-1)^{-1}\,0 = xD^{-1}x^{-2}\,0 = Cx = xf(y),$$

by (*d*) and by integration. But this is equivalent to

$$\frac{d^2u}{dx^2} - \frac{d^2u}{dy^2} = xf(y).$$

Make $u = v - xD_y^{-2}f(y)$, and we have $\dfrac{d^2v}{dx^2} - \dfrac{d^2v}{dy^2} = 0.$

Consequently $\; v = \phi\,(y+x) + \psi\,(y-x),\; z = (xD_x-1)\,u$

$$= (xD_x-1)\,v - (xD_x-1)\,xD_y^{-2}f(y) = (xD_x-1)\,v = x\,\frac{dv}{dx} - v\,;$$

or $z = x\,\{\phi'\,(y+x) - \psi'\,(y-x)\} - \{\phi\,(y+x) + \psi\,(y-x)\}.$

If we had not operated upon the cypher, we should have obtained the same result; but we should not have had the same assurance of the generality of the solution.

I presume this method may be applied to similar equations in finite differences.

Gunthwaite Hall, Penistone, March, 1846.

ON THE THEORY OF MAGIC SQUARES, CUBES, &C.

By R. Moon, M.A., Fellow of Queens' College.

In a former paper, published in this *Journal*, I endeavoured to develope a new method of treating the subject of Magic Squares, and exhibited more or less fully the mode of its application to the case of squares containing an odd number of places. On the present occasion I purpose to shew that the same method may be applied to the composition of Magic Cubes, and I shall in conclusion say a few words on the extension of the theory to squares of even numbers.

For the sake of simplicity I shall confine myself to the cube made up of the natural numbers from 0 to 26, both inclusive; which may be derived from the formula

$$x + 3y + 3^2 z,$$

by giving successively to xyz the values 0.1.2 respectively.

The numbers represented by the nine following columns properly arranged, *i.e.* the second line being placed behind the first and the third behind the second, will form a magic cube; except as regards the diagonals, to which I shall afterwards direct attention.

A	*B*	*C*
$x_0 + 3y_0 + 3^2 z_0$	$x_1 + 3y_2 + 3^2 z_1$	$x_2 + 3y_1 + 3^2 z_2$
$x_1 + 3y_1 + 3^2 z_1$	$x_2 + 3y_0 + 3^2 z_2$	$x_0 + 3y_2 + 3^2 z_0$
$x_2 + 3y_2 + 3^2 z_2$	$x_0 + 3y_1 + 3^2 z_0$	$x_1 + 3y_0 + 3^2 z_1$

D	*E*	*F*
$x_1 + 3y_1 + 3^2 z_2$	$x_2 + 3y_0 + 3^2 z_0$	$x_0 + 3y_2 + 3^2 z_1$
$x_2 + 3y_2 + 3^2 z_0$	$x_0 + 3y_1 + 3^2 z_1$	$x_1 + 3y_0 + 3^2 z_2$
$x_0 + 3y_0 + 3^2 z_1$	$x_1 + 3y_2 + 3^2 z_2$	$x_2 + 3y_1 + 3^2 z_0$

G	*H*	*K*
$x_2 + 3y_2 + 3^2 z_1$	$x_0 + 3y_1 + 3^2 z_2$	$x_1 + 3y_0 + 3^2 z_0$
$x_0 + 3y_0 + 3^2 z_2$	$x_1 + 3y_2 + 3^2 z_0$	$x_2 + 3y_1 + 3^2 z_1$
$x_1 + 3y_1 + 3^2 z_0$	$x_2 + 3y_0 + 3^2 z_1$	$x_0 + 3y_2 + 3^2 z_2$

from each other according to the order of their indices: thus in the column A the indices of x are 0.1.2, in D they are 1.2.0, but though the initial figure is different, the figures occur *in the same order;* and the same holds of the other columns.

Again, B is formed from A, as regards the x's and z's, by rejecting the head of each column and throwing it to the base: the column of y's is formed by elevating the base to the top: C is formed from B in the same manner as B from A, and E, F: H, K are formed from D and G respectively in the same manner as B and C are formed from A.

D is formed from A by treating the x's and y's in exactly the same way as the x's and z's are treated in the formation of B from A, and treating the z's in the former case as y in the latter; which we may, if we choose, express by saying that y and z are to be interchanged: but in practice it is better not to adopt this latter view.

G is formed from D in the same way as D from A; and $E\ H$, $F\ K$ respectively might be formed from B and C respectively in a similar manner.

With regard to the number of different cubes to be obtained by the above method, I would observe that in the above example we may interchange at pleasure z_0 and z_1, z_1 and z_2, z_2 and z_0, so that if n be the number of cubes we should obtain independently of this consideration, we may by means of it increase the number to $3n$. Also we may interchange y_0 and y_2 (but not $y_1 . y_2 : y_1 . y_0$, as is easily seen from the consideration of the diagonals), and we may interchange x_0 and x_2, so that on the whole we shall obtain 2.2.3 cubes. But it is evident that in the above method z is, so to speak, the centre of the system, and as x and y have each equal claims in that respect the entire number of different cubes is $2^2.3^2$ or $1^2.2^2.3^2$.

It will be perceived that any one vertical row may be changed for another: thus we may interchange ADG with CFH, and so on; and in like manner any one horizontal row may be changed for another. The only effect of these changes will be on the diagonals of the cube, which however may always be adjusted by properly assuming the initial column.

The next remark I shall make bears on the subject of magic squares, as well as on that of magic cubes. The above method applies independently of the absolute values of $x_0y_0z_0$, $x_1y_1z_1$, $x_2y_2z_2$, provided only that, as regards one class of those quantities, the x's for example, one of the three values $x_0x_1x_2$ is the mean of the other two. Hence any series

The next is composed of the numbers from 0 to 63 inclusive.

(1)	(2)	(3)	(4)
0 + 6	5 + 1	4 + 6	1 + 1
7 + 2	2 + 5	3 + 2	6 + 5
0 + 4	5 + 3	4 + 4	1 + 3
7 + 0	2 + 7	3 + 0	6 + 7
7 + 7	2 + 0	3 + 7	6 + 0
1 + 3	5 + 4	4 + 3	1 + 4
7 + 5	2 + 2	3 + 5	6 + 2
1 + 1	5 + 6	4 + 1	1 + 6

(5)	(6)	(7)	(8)
6 + 1	3 + 6	2 + 1	7 + 6
1 + 5	4 + 2	5 + 5	0 + 2
6 + 3	3 + 4	2 + 3	7 + 4
1 + 7	4 + 0	5 + 7	0 + 0
1 + 0	4 + 7	5 + 0	0 + 7
6 + 4	3 + 3	2 + 4	7 + 3
1 + 2	4 + 5	5 + 2	0 + 5
6 + 6	3 + 1	2 + 6	7 + 1.

I may observe that in my previous paper the number of magic squares, produced by the methods there indicated, is considerably under-estimated. It is there stated that in the case of a square of twenty-five places the effect produced by rejecting the *three* first x's from the top of the column would be to give the same result as would be obtained by rejecting the *two* first, but in the reverse order, which is not the fact. The squares obtained on the former principle of formation are distinct from those composed in the latter. Also the number of squares may be increased by combining the principle of the two methods explained in the paper alluded to. Thus the column of x's may be formed by rejecting the first, and that of the y's by rejecting the two first members (the initial x being always the mean value), and *vice versâ.*

Liverpool, December 29, 1845.

ON THE GEOMETRICAL REPRESENTATION OF THE MOTION OF A SOLID BODY.

By Arthur Cayley.

Let $P, Q, R, \ldots.$ be consecutive generating lines of a skew surface, and on these take points p', p; q', q; $r', r \ldots.$ such that $pq', qr' \ldots.$ are the shortest distances between P and Q, Q and R, &c. Then for the generating line P, the ratio of

the torsion. [This includes also the case in which one surface is a transformation of the other, where the motion is evidently a rolling one.] A skew surface moving in this manner upon another of which it is the deformation, may be said to "glide" upon it. We may now state the kinematical theorem.

"Any motion whatever of a solid body in space may be represented as the 'gliding' motion of one skew surface upon another fixed in space, and of which it is the deformation."

A theorem which is to be considered as the generalization of the well known one—

"Any motion of a solid body round a fixed point may be represented as the rolling motion of a conical surface upon a second conic surface fixed in space."

And of the supplementary theorem—

"The angular velocity round the line of contact (the instantaneous axis) is to the angular velocity of this line as the difference of curvatures of the two cones at any point in the same line, to the reciprocal of the distance of the point from the vertex."

The analytical demonstration of this last theorem is rather interesting: it depends on the following formulæ. Forming two determinants, the first with the angular velocities round three axes fixed in space, and the first and second derived coefficients with respect to the time of these velocities; the other in the same way with the angular velocities round axes fixed in the body; the difference of these determinants is equal to the fourth power of the angular velocity into the square of the angular velocity of the instantaneous axis.

To show this, let p, q, r be the angular velocities round the axes fixed in the body; u, v, w those round axes fixed in space; ω the angular velocity round the instantaneous axis; ∇, Ω the two determinants: the theorem comes to

$$\nabla - \Omega = M,$$

where $M = \omega^2 (p'^2 + q'^2 + r'^2 - \omega'^2)$, or $\omega^2 (u'^2 + v'^2 + w'^2 - \omega'^2)$.

Here
$$\begin{aligned} u &= \alpha p + \beta q + \gamma r, \\ v &= \alpha' p + \beta' q + \gamma' r, \\ w &= \alpha'' p + \beta'' q + \gamma'' r. \end{aligned}$$

Whence
$$\begin{aligned} u' &= \alpha p' + \beta q' + \gamma r', \\ v' &= \alpha' p' + \beta' q' + \gamma' r', \\ w' &= \alpha'' p' + \beta'' q' + \gamma'' r', \end{aligned}$$

these objections. Imagine two sets of axes Ax, Ay, Az, $Ax_{\prime}$, $Ay_{\prime}$, $Az_{\prime}$. The former set can be made to coincide with the second set, by a rotation θ round a certain axis AR, inclined to Ax, Ay, Az at angles f, g, h. (As usual f, g, h are the angles RAx, RAy, RAz considered as positive, and the rotation is in the same direction as a rotation round Az from x towards y). This axis may be termed the resultant axis, and the angle θ the resultant rotation. The formulæ of Euler express the coefficients of the transformation in terms of the resultant rotation and of the position of the resultant axis, *i.e.* in terms of θ and of the angles f, g, h, whose cosines are connected by the equation

$$\cos^2 f + \cos^2 g + \cos^2 h = 1.$$

This idea was improved upon by M. Rodrigues (Liouv. tom. v. p. 404), who introduced the quantities

$$\tan \tfrac{1}{2}\theta \cos f, \quad \tan \tfrac{1}{2}\theta \cos g, \quad \tan \tfrac{1}{2}\theta \cos h,$$

(quantities which will be represented by λ, μ, ν,) by means of which he expressed the coefficients as fractions, the numerators of which are very simple rational functions of the second order of λ, μ, ν, and which have the common denominator $(1 + \lambda^2 + \mu^2 + \nu^2)$. These quantities may conveniently be termed the "coordinates of the resultant rotation," and the denominator or the square of the secant of the semiangle of resultant rotation will be the "modulus" of the rotation. The elegance of these results led me to apply them to the mechanical question, and I gave in the *Journal* (vol. III. p. 224) the differential equations of motion obtained in terms of λ, μ, ν: which I integrated as in the common theory, by supposing one of the fixed axes to be perpendicular to the invariable plane. Though my attention was again called to the subject, by the connexion of some of these formulæ with Sir William Hamilton's theory of quaternions, no other way of performing the integration occurred to me. The grand discovery however of Jacobi, of the possibility of reducing to quadratures the two final differential equations of any mechanical problem, when the remaining integrals are known, induced me to resume the problem, and at least attempt to bring it so far as to obtain a differential equation of the first order between two variables only, the multiplier of which could be obtained theoretically by Jacobi's discovery. The choice of two new variables to which the equations of the problem led me, enabled me to effect this with the greatest simplicity; and the differential equation which I finally obtained, turned out

$$\left.\begin{aligned}\Lambda &= \tfrac{1}{2}\left\{(1+\lambda^2)p + (\lambda\mu - \nu)q + (\lambda\nu + \mu)r\right\}\\ \mathrm{M} &= \tfrac{1}{2}\left\{(\mu\lambda + \nu)p + (1+\mu^2)q + (\mu\nu - \lambda)r\right\}\\ \mathrm{N} &= \tfrac{1}{2}\left\{(\nu\lambda - \mu)p + (\mu\nu + \lambda)q + (1+\nu^2)r\right\}\end{aligned}\right\}\ldots\ldots(3).$$

And in the case where the forces vanish, the first three equations become simply

$$\left.\begin{aligned}P &= \frac{1}{A}(B-C)qr,\\ Q &= \frac{1}{B}(C-A)rp,\\ R &= \frac{1}{C}(A-B)pq.\end{aligned}\right\}\ldots\ldots\ldots\ldots\ (4).$$

In which case the usual four integrals of the system are

$$Ap^2 + Bq^2 + Cr^2 = h\ \ldots\ldots\ldots\ldots\ (5),$$

$$\left.\begin{aligned}&Ap(1+\lambda^2-\mu^2-\nu^2) + 2Bq(\lambda\mu-\nu) + 2Cr(\nu\lambda+\mu) = \mathrm{a}(1+\lambda^2+\mu^2+\nu^2)\\ &2Ap(\lambda\mu+\nu) + Bq(1+\mu^2-\nu^2-\lambda^2) + 2Cr(\mu\nu-\lambda) = \mathrm{b}(1+\lambda^2+\mu^2+\nu^2)\\ &2Ap(\nu\lambda-\mu) + 2Bq(\mu\nu+\lambda) + Cr(1+\nu^2-\lambda^2-\mu^2) = \mathrm{c}(1+\lambda^2+\mu^2+\nu^2)\end{aligned}\right\}\ldots\ldots\ (6).$$

Or as they may also be written,

$$\left.\begin{aligned}&\mathrm{a}(1+\lambda^2-\mu^2-\nu^2) + 2\mathrm{b}(\lambda\mu+\nu) + 2\mathrm{c}(\nu\lambda-\mu) = Ap(1+\lambda^2+\mu^2+\nu^2)\\ &2\mathrm{a}(\lambda\mu-\nu) + \mathrm{b}(1+\mu^2-\nu^2-\lambda^2) + 2\mathrm{c}(\mu\nu+\lambda) = Bq(1+\lambda^2+\mu^2+\nu^2)\\ &2\mathrm{a}(\nu\lambda+\mu) + 2\mathrm{b}(\mu\nu-\lambda) + \mathrm{c}(1+\nu^2-\lambda^2-\mu^2) = Cr(1+\lambda^2+\mu^2+\nu^2)\end{aligned}\right\}\ldots\ldots\ (6\ \textit{bis}).$$

To which we may add,

$$A^2p^2 + B^2q^2 + C^2r^2 = k^2\ldots\ldots\ldots\ldots\ldots(7);$$

where
$$k^2 = \mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Introducing the quantities κ, Ω, (the former of which has been already made use of) given by the equations

$$\left.\begin{aligned}\kappa &= 1 + \lambda^2 + \mu^2 + \nu^2,\\ \Omega &= \lambda Ap + \mu Bq + \nu Cr\end{aligned}\right\}\ldots\ldots\ldots\ldots\ (9).$$

The equations (6) may be written under the form

$$\left.\begin{aligned}2\lambda\Omega + 2\mu Cr - 2\nu Bq &= \kappa(Ap + \mathrm{a}) - 2Ap\\ -2\lambda Cr + 2\mu\Omega + 2\nu Ap &= \kappa(Bq + \mathrm{b}) - 2Bq\\ 2\lambda Bq - 2\mu Ap + 2\nu\Omega &= \kappa(Cr + \mathrm{c}) - 2Cr\end{aligned}\right\}\ldots(10).$$

Whence also, multiplying by Ap, Bq, Cr, and adding,

$$2\Omega^2 = \kappa\left\{k^2 + (Ap\mathrm{a} + Bq\mathrm{b} + Cr\mathrm{c})\right\} - 2k^2\ldots(11),$$

or from the equations (1), (3),

$$d\kappa = \kappa\,(\lambda p + \mu q + \nu r)\,dt \qquad (22).$$

Whence, from (16),

$$2\upsilon d\kappa = \kappa\left\{\Omega\,(h + \Phi) - \nabla\right\} dt \qquad (23);$$

or

$$2\,(\upsilon d\kappa + \kappa d\upsilon) = \kappa\Omega\,(h + \Phi)\,dt \qquad (24).$$

Whence

$$d\Omega = \tfrac{1}{4}\,\kappa\,(h + \Phi)\,dt$$

$$= \tfrac{1}{4}\,\frac{\Omega^2 + k^2}{\upsilon}\,(h + \Phi)\,dt \qquad (25).$$

And therefore, from (19),

$$\frac{2d\Omega}{\Omega^2 + k^2} = \frac{h + \Phi}{\upsilon\nabla}\,d\upsilon \qquad (26),$$

the required differential equation, in which Φ, ∇ are given functions of (υ), *i.e.* they are functions of p, q, r by the equations (15), and these quantities are functions of υ by (18). The variables in (26) are therefore separated, and we have the integral equation

$$2\tan^{-1}\frac{\Omega}{k} = \delta + k\int\frac{(h + \Phi)\,d\upsilon}{\upsilon\nabla} \qquad (27),$$

where δ is the constant of integration. The equation (19) gives also

$$t - \epsilon = 2\int\frac{d\upsilon}{\nabla} \qquad (28);$$

and thus the solution of the problem is completely effected. The integrals may be taken from any particular value υ_0 of υ. The variable Ω may be exhibited as the integral of an *explicit* algebraical function, by recurring to the variable ϕ of the paper quoted.

Thus if

$$Ap_0^2 + Bq_0^2 + Cr_0^2 = h,$$

$$A^2p_0^2 + B^2q_0^2 + C^2r_0^2 = k^2,$$

$$Ap_0\mathrm{a} + Bq_0\mathrm{b} + Cr_0\mathrm{c} = 2\upsilon_0 - k^2;$$

then

$$\sqrt{\left\{p_0^2 - \frac{1}{A}\,(C - B)\,\phi\right\}},\quad \sqrt{\left\{q_0^2 - \frac{1}{A}\,(A - C)\,\phi\right\}}$$

$$\sqrt{\left\{r_0^2 - \frac{1}{C}\,(B - A)\,\phi\right\}},$$

$$dt = \tfrac{1}{2}\,\frac{d\phi}{pqr} = \frac{2d\upsilon}{\nabla},\quad \text{or}\quad \frac{d\upsilon}{\nabla} = \tfrac{1}{4}\,\frac{d\phi}{pqr};$$

Formulæ for the variation of the arbitrary constants, in the case of any distributing forces acting upon the body, will be given in a subsequent paper.

[redacted]

other by indefinitely small distances'; if we add to this general notion, the assertion, that '*these molecules act on each other only in the line joining them,*' we shall have a definition of the medium whose laws I propose to investigate. Suppose now that this medium is acted on by *no external* forces, and abandoned solely to the action of its molecules on each other; the distinction which I conceive to exist between solids and fluids is the following:

That in solid bodies the resultant of all the forces exerted by all the surrounding molecules on any molecule (m), *is zero. That in fluids, whether liquid or gaseous, this is not the case, and that consequently the fluid (no external pressures or forces acting) would be dissipated.*

That this is the correct distinction of these two classes of bodies, will, I hope, be made clear by the following investigation, and at present it will be sufficient to observe that it agrees with the ordinary notion of a fluid: in such a body we suppose a pressure (p) to exist at each point (x, y, z), which equilibrates the external forces, such as the forces arising from the sides of the vessel containing the fluid, &c. Hence; if the *external forces cease to act*, the pressure being transmitted to the external surface of the fluid would dissipate it.

The case of a fluid in a closed vessel is not the case here considered, for the pressures of the sides of the vessel are, in this case, the external forces, which, together with the molecular forces, produce equilibrium: but the molecular forces themselves are not, I conceive, in equilibrium. This is manifestly true of gases, and I consider the same to be true of liquids. The distinction between liquids and gases is, probably, *relative* to the ordinary external forces in action at the surface of the earth, such as gravity, the pressure of the atmosphere, &c.

I proceed, without further delay, to the laws of the medium, whether solid or fluid.

The general equation of equilibrium of the points composing a medium is

$$\iiint(X\delta\xi + Y\delta\eta + Z\delta\zeta)\,dm = \iiint\delta V\,dx\,dy\,dz \;\ldots.\; (1),$$

in which equation the left-hand member is the sum of the '*moments*' of the external forces, and the right-hand member is the sum of the *moments* of the internal forces. I use the term *moments* as defined by Lagrange.

The function V depends, in general, on the particular nature of the medium considered, and its form must, in the present case, be deduced from the definition already given. x, y, z are the coordinates of the position of rest of any

molecule (m); and $x+\xi$, $y+\eta$, $z+\zeta$ are the coordinates of the same molecule when displaced by the action of external forces.

When the molecules are in a state of rest, not acted upon by any external forces, the force exerted by any molecule (m') on (m) will be, in general, a function of the *distance* (mm'), and of the *direction* of the line (mm'): if now, by means of external forces, the molecules $(m, m', \&c.)$ assume new positions, the force exerted by m' on m will be represented in general by the expression

$$f(\rho, \alpha, \beta, \gamma, \rho'),$$

$\rho, \alpha, \beta, \gamma$ being the *original* length and direction of the line (mm'), and ρ' being the alteration in ρ; ρ' being small, the function (f) may be represented by

$$f = F_0 + 2F_1\rho' + 3F_2\rho'^2 + \&c. \dots\dots\dots (2),$$

$F_0, F_1, \&c.$ being functions of $(\rho, \alpha, \beta, \gamma)$.

As the force (f) is a force tending to alter $\rho+\rho'$, its *moment* will be $f.\delta\rho'$, or neglecting ρ'^2, &c.

$$F_0\delta\rho' + 2F_1\rho'\delta\rho'.$$

Hence
$$\delta V = \Sigma \{F_0\delta\rho' + F_1\delta(\rho'^2)\};$$
and therefore

$$V = \int_0^\infty \int_0^\pi \int_0^{2\pi} (F_0\rho' + F_1\rho'^2)\, \rho^2 \sin\theta\, d\rho\, d\theta\, d\phi \dots. (3).$$

The value of (ρ'), to be substituted in this expression for V, is thus found. Let $x, y, z, x+a, y+b, z+c$, be the coordinates of rest of m and m'; then in the changed position, if x, y, z, become $x+\xi$, $y+\eta$, $z+\zeta$, the coordinates of m' will become

$$\left.\begin{array}{l} x + \xi + a + \dfrac{d\xi}{dx}a + \dfrac{d\xi}{dy}b + \dfrac{d\xi}{dz}c, \\[2ex] y + \eta + b + \dfrac{d\eta}{dx}a + \dfrac{d\eta}{dy}b + \dfrac{d\eta}{dz}c, \\[2ex] z + \zeta + c + \dfrac{d\zeta}{dx}a + \dfrac{d\zeta}{dy}b + \dfrac{d\zeta}{dz}c, \end{array}\right\}.$$

$\rho+\rho'$ is equal to the square root of the sum of the squares of the differences of the coordinates of its extreme points, *i.e.*

$$\rho+\rho' = \sqrt{\left\{\left(a + \frac{d\xi}{dx}a + \frac{d\xi}{dy}b + \frac{d\xi}{dz}c\right)^2 + \left(b + \frac{d\eta}{dx}a + \frac{d\eta}{dy}b + \frac{d\eta}{dz}c\right)^2 \right.}$$
$$\left. + \left(c + \frac{d\zeta}{dx}a + \frac{d\zeta}{dy}b + \frac{d\zeta}{dz}c\right)^2\right\};$$

and, neglecting the smaller quantities, we obtain

$$\rho + \rho' = \rho + \rho\left(\frac{d\xi}{dx}\cos^2 \alpha + \frac{d\eta}{dy}\cos^2 \beta + \frac{d\zeta}{dz}\cos^2 \gamma \right.$$

$$\left. + u \cos \beta \cos \gamma + v \cos \alpha \cos \gamma + w \cos \alpha \cos \beta\right),$$

where

$$u = \frac{d\eta}{dz} + \frac{d\zeta}{dy},$$

$$v = \frac{d\zeta}{dx} + \frac{d\xi}{dz},$$

$$w = \frac{d\xi}{dy} + \frac{d\eta}{dx};$$

and also

$$\rho = \sqrt{(a^2 + b^2 + c^2)}.$$

Hence finally we obtain

$$\rho' = \rho\left\{\frac{d\xi}{dx}\cos^2 \alpha + \frac{d\eta}{dy}\cos^2 \beta + \frac{d\zeta}{dz}\cos^2 \gamma + \left(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\right)\cos\beta\cos\gamma \right.$$

$$\left. + \left(\frac{d\zeta}{dx} + \frac{d\xi}{dz}\right)\cos\alpha\cos\gamma + \left(\frac{d\xi}{dy} + \frac{d\eta}{dx}\right)\cos\alpha\cos\beta\right\}.$$

This is the value of ρ', to be substituted in V, which will then consist of two parts, V_0 and V_1, depending upon F_0 and F_1, V_0 being homogeneous and linear with respect to

$$\frac{d\xi}{dx},\ \frac{d\eta}{dy},\ \frac{d\zeta}{dz},\ u,\ v,\ w;$$

and V_1 homogeneous, and of the second order with respect to the same six quantities.

Returning now to the distinction drawn between solids and fluids, it will appear from that distinction that in fluids the function V will be $V_0 + V_1$, while for solids $V_0 = 0$: for if in (2) we suppose $\rho' = 0$, we shall have $F_0 = f$; hence the definition of a solid requires that the forces F_0 which correspond to the case of *no external forces* acting should equilibrate.

I shall now determine the equations of equilibrium arising from V_0, which does not vanish for fluids, and then examine the case of solids; for which $V_0 = 0$, and which depend only on V_1.

By formula (3), we have

$$V_0 = \int_0^\infty\int_0^\pi\int_0^{2\pi} F_0\rho'.\rho^2 \sin\theta\, d\rho\, d\theta\, d\phi.$$

which are the well-known equations of Hydrostatics. Hence, if the forces (X, Y, Z) be zero at all points of the fluid, the quantity p must be constant; and vice versâ, if p be constant for all points of the fluid, the forces (X, Y, Z) must be zero—(this includes the case of homogeneous fluids): in such a case the function V_0 will give only the condition at the limits expressed by the double integrals (6); which is identical with that found by Lagrange, and expresses that there must be a normal pressure at every point of the bounding surface, constant and equal to p, in order that the fluid should remain in equilibrium. This condition at the limits is not necessary in solids, since for them V_0, and therefore p, is equal to zero.

It follows from the views I have adopted in this paper, that the ordinary equations of Hydrostatics and Hydrodynamics are only a first approximation to the whole equations, and that in some cases, particularly in Hydrodynamics, this approximation may be insufficient: in such cases we should add to the equations (7) the terms arising from V_1, which are common to solids and fluids.

In general $$V = V_0 + V_1 + V_2 + \&c.,$$

and the terms V_0, V_1, &c. will give rise, in the equations of equilibrium, to differential coefficients of the first, second, &c. order.

I shall now take up the general discussion of the equations of equilibrium and motion arising from the function (V_1).

From equation (3) we see that

$$V_1 = \int_0^\infty \int_0^\pi \int_0^{2\pi} F_1 \cdot (\rho')^2 \cdot \rho^2 \sin\theta d\rho d\theta d\phi.$$

Hence, making $d\omega = \rho^2 \sin\theta\, d\rho\, d\theta\, d\phi$, we shall have

$$V_1 = \int_0^\infty \int_0^\pi \int_0^{2\pi} F_1 \left(\frac{d\xi}{dx}\cos^2\alpha + \frac{d\eta}{dy}\cos^2\beta + \frac{d\zeta}{dz}\cos^2\gamma\right.$$
$$\left. + u\cos\beta\cos\gamma + v\cos\alpha\cos\gamma + w\cos\alpha\cos\beta\right)^2 \rho^2 d\omega \ldots.(8).$$

Therefore

$$2V_1 = \left\{A\left(\frac{d\xi}{dx}\right)^2 + B\left(\frac{d\eta}{dy}\right)^2 + C\left(\frac{d\zeta}{dz}\right)^2\right\} + \{Lu^2 + Mv^2 + Nw^2\}$$
$$+ 2\left(L\frac{d\eta}{dy}\cdot\frac{d\zeta}{dz} + M\frac{d\xi}{dx}\cdot\frac{d\zeta}{dz} + N\frac{d\xi}{dx}\cdot\frac{d\eta}{dy}\right)$$
$$+ 2(\alpha_1 vw + \beta_2 uw + \gamma_3 uv)$$

$$-\iiint\left\{\left(M\frac{dv}{dz}+N\frac{dw}{dy}\right)\delta\xi+\left(N\frac{dw}{dx}+L\frac{du}{dz}\right)\delta\eta\right.$$
$$\left.+\left(L\frac{du}{dy}+M\frac{dv}{dx}\right)\delta\zeta\right\}dxdydz.$$

(c) $$\tfrac{1}{2}\delta\iiint 2\left(L\frac{d\eta}{dy}\cdot\frac{d\zeta}{dz}+M\frac{d\xi}{dx}\cdot\frac{d\zeta}{dz}+N\frac{d\xi}{dx}\cdot\frac{d\eta}{dy}\right)dxdydz$$
$$=\iint\left(M\frac{d\zeta}{dz}+N\frac{d\eta}{dy}\right)\delta\xi dydz$$
$$+\iint\left(N\frac{d\xi}{dx}+L\frac{d\zeta}{dz}\right)\delta\eta dxdz$$
$$+\iint\left(L\frac{d\eta}{dy}+M\frac{d\xi}{dx}\right)\delta\zeta dxdy$$
$$-\iiint\left\{\left(M\frac{d^2\zeta}{dxdz}+N\frac{d^2\eta}{dxdy}\right)\delta\xi+\left(N\frac{d^2\xi}{dxdy}+L\frac{d^2\zeta}{dydz}\right)\delta\eta\right.$$
$$\left.+\left(L\frac{d^2\eta}{dydz}+M\frac{d^2\xi}{dxdz}\right)\delta\zeta\right\}dxdydz.$$

(d) $$\tfrac{1}{2}\delta\iiint 2(\alpha_1 vw+\beta_2 uw+\gamma_3 uv)\,dxdydz$$
$$=\iint\{\alpha_1(v\delta\eta+w\delta\zeta)+u(\beta_2\delta\eta+\gamma_3\delta\zeta)\}\,dydz$$
$$+\iint\{\beta_2(w\delta\zeta+u\delta\xi)+v(\gamma_3\delta\zeta+\alpha_1\delta\xi)\}\,dxdz$$
$$+\iint\{\gamma_3(u\delta\xi+v\delta\eta)+w(\alpha_1\delta\xi+\beta_2\delta\eta)\}\,dxdy.$$
$$-\iiint\left\{\alpha_1\left(\frac{dv}{dy}+\frac{dw}{dz}\right)+\beta_2\frac{du}{dy}+\gamma_3\frac{du}{dz}\right\}\delta\xi\,dxdydz$$
$$-\iiint\left\{\beta_2\left(\frac{dw}{dz}+\frac{du}{dx}\right)+\gamma_3\frac{dv}{dz}+\alpha_1\frac{dv}{dx}\right\}\delta\eta\,dxdydz$$
$$-\iiint\left\{\gamma_3\left(\frac{du}{dx}+\frac{dv}{dy}\right)+\alpha_1\frac{dw}{dx}+\beta_2\frac{dw}{dy}\right\}\delta\zeta dxdydz.$$

If we assume

$$\aleph=\alpha_1\frac{d\xi}{dx}+\beta_1\frac{d\eta}{dy}+\gamma_1\frac{d\zeta}{dz},$$
$$\beth=\alpha_2\frac{d\xi}{dx}+\beta_2\frac{d\eta}{dy}+\gamma_2\frac{d\zeta}{dz},$$
$$\gimel=\alpha_3\frac{d\xi}{dx}+\beta_3\frac{d\eta}{dy}+\gamma_3\frac{d\zeta}{dz},$$

$$R = C\frac{d^2\zeta}{dz^2} + M\frac{d^2\zeta}{dx^2} + L\frac{d^2\zeta}{dy^2} + 2\left(\gamma_3\frac{d^2\zeta}{dxdy} + \gamma_1\frac{d^2\zeta}{dydz} + \gamma_2\frac{d^2\zeta}{dxdz}\right)$$

$$+ \alpha_2\frac{d^2\xi}{dx^2} + \beta_2\frac{d^2\xi}{dy^2} + \gamma_2\frac{d^2\xi}{dz^2} + 2\left(\gamma_3\frac{d^2\xi}{dydz} + M\frac{d^2\xi}{dxdz} + \alpha_1\frac{d^2\xi}{dxdy}\right)$$

$$+ \alpha_1\frac{d^2\eta}{dx^2} + \beta_1\frac{d^2\eta}{dy^2} + \gamma_1\frac{d^2\eta}{dz^2} + 2\left(\gamma_3\frac{d^2\eta}{dxdz} + L\frac{d^2\eta}{dydz} + \beta_2\frac{d^2\eta}{dxdy}\right).$$

The general equation of equilibrium and motion, corresponding to the function V_1, will be

$$\iiint \varepsilon\,(X\delta\xi + Y\delta\eta + Z\delta\zeta)\,dxdydz$$
$$= \Delta - \iiint (P\delta\xi + Q\delta\eta + R\delta\zeta)\,dxdydz,$$

where $dm = \varepsilon dxdydz$; therefore the equations of equilibrium are

$$-\varepsilon X = P, \quad -\varepsilon Y = Q, \quad -\varepsilon Z = R. \ldots\ldots(11).$$

These equations are the equations of equilibrium of a solid body expressed in their most general form, without making any supposition as to the arrangement of the molecules in the body, and supposing the force in action between any two molecules to be a function, as well of the *direction* of the line joining them, as of the *length* of that line; which is the most general conception of a crystalline structure.

Note. As the continuation of this paper will not appear until the next No. of the *Cambridge and Dublin Mathematical Journal*, I shall here mention some of the results at which I have arrived. Having simplified the function V_1, I have integrated the equations of motion by means of a particular integral, which is general enough to give, by means of the relations among the constants, many geometrical properties of the motion of elastic solids. In examining the propagation of waves through the medium, I have used the *surface of wave-slowness*, which is of the sixth degree, and possesses nodes in its principal planes, which give rise to a theory of conical refraction of the vibrations of solids, somewhat analogous to the corresponding case of light. In the case of homogeneous, uncrystalline bodies, the whole theory becomes exceedingly simple.

Trinity College, Dublin, March, 1846.

(*To be continued.*)

[redacted]

being in, and the latter perpendicular to, the plane of refraction; a, b, c the optical constants referring to them, that is, according to Fresnel's theory, the velocities of propagation of waves in which the vibrations are parallel to the three axes respectively. Everything being symmetrical with respect to the plane of incidence, we need only consider what takes place in that plane. This plane will cut the wave surface in a circle of radius c, and an ellipse whose semiaxes are a along oB and b along oA. We have only got to consider the ellipse, since it is it that determines the direction of the extraordinary ray. The form of the crystal will very often make known the directions of the axes of elasticity. Supposing these directions known, let α, β denote the inclinations of oA, oB to the produced parts of EA, EB respectively; α, β and i being of course connected by the equation $\alpha + \beta = \frac{\pi}{2} + i$.

Let ϕ, ψ be the angles of incidence and emergence, the light being supposed incident on the face EA; ϕ' the inclination of the refracted wave to EA, ψ' its inclination to EB, D the deviation, v the velocity of the wave within the crystal, u its velocity in the outer medium, which may be supposed to be either air, or a liquid of known refractive power. Then we have

$$D = \phi + \psi - i\,^{*} \ldots\ldots\ldots\ldots\ldots (1),$$

$$\phi' + \psi' = i \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$v \sin \phi = u \sin \phi' \ldots\ldots\ldots\ldots\ldots (3),$$

$$v \sin \psi = u \sin \psi' \ldots\ldots\ldots\ldots\ldots (4),$$

$$v^2 = a^2 \cos^2 (\alpha - \phi') + b^2 \sin^2 (\alpha - \phi') \ldots\ldots (5).$$

From (2), (3), (4),

$$u \sin \psi' = v \sin \psi = u \sin (i - \phi') = u \sin i \cos \phi' - v \cos i \sin \phi;$$

$$\therefore \quad \cos \phi' = \frac{v}{u \sin i} (\sin \psi + \cos i \sin \phi);$$

and
$$\sin \phi' = \frac{v}{u \sin i} \sin i \sin \phi:$$

substituting in (5),

$$u^2 \sin^2 i = a^2 \left\{\cos \alpha (\sin \psi + \cos i \sin \phi) + \sin \alpha \sin i \sin \phi\right\}^2$$
$$+ b^2 \left\{\sin \alpha (\sin \psi + \cos i \sin \phi) - \cos \alpha \sin i \sin \phi\right\}^2,$$

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* I am indebted to the Rev. P. Frost for the suggestion of employing equations (1)....(4), rather than making use of the ellipse in which the wave surface is cut by the plane of incidence.

Remarque. Le triangle dont nous venons de parler, joue, comme on voit, un rôle assez important dans la théorie de la lemniscate, aussi je ne crois pas inutile de mentionner une dernière propriété, qui consiste en ce que l'aire de ce triangle et l'aire du secteur de la courbe ont la même différentielle.

§. III. Soit maintenant n un nombre entier ou fractionnaire, ou même incommensurable et construisons le triangle OMP tel que

$$OP = \sqrt{n} \quad \text{et} \quad MP = \sqrt{(n+1)}.$$

Puis imaginons que le sommet O restant fixe, le triangle varie de telle sorte que le cosinus de l'angle θ formé par le seul coté variable OM avec une droite fixe, soit constamment égal au cosinus de l'angle

$$n.MOP - (n+1)\, OMP,$$

le point M engendrera une courbe (algébrique si n est commensurable) dont l'arc sera une fonction elliptique du rayon vecteur, réductible au module $\sqrt{\left(\frac{n}{n+1}\right)}$,

Soit en effet $MOP = \alpha$, $OMP = \beta$, l'équation de la courbe résultera de l'éliminations de α et β entre

$$\cos\theta = \cos\{n\alpha - (n+1)\beta\}$$

$$\left\{\begin{aligned} \cos\alpha &= \frac{r^2-1}{2r\sqrt{n}}, \\ \cos\beta &= \frac{r^2+1}{2r\sqrt{(n+1)}}, \end{aligned}\right. \quad \text{d'où} \quad \left\{\begin{aligned} \sin\alpha &= \frac{\Delta}{2r\sqrt{n}}, \\ \sin\beta &= \frac{\Delta}{2r\sqrt{(n+1)}}, \end{aligned}\right.$$

en faisant pour abréger

$$\Delta = \sqrt{\{-r^4 + 2(2n+1)r^2 - 1\}}.$$

cela posé on déduit par la différentiation

$$\pm\, d\theta = n d\alpha - (n+1)\, d\beta,$$

et
$$d\alpha = -\frac{r^2+1}{\Delta}\,\frac{dr}{r}, \quad d\beta = -\frac{r^2-1}{\Delta}\,\frac{dr}{r},$$

d'où
$$d\theta = \frac{r^2 - (2n+1)}{\Delta}\,\frac{dr}{r},$$

et par suite
$$ds = 2\sqrt{\{n(n+1)}\,\frac{dr}{\Delta}.$$

Des équations précédentes on déduit encore les formules suivantes qu'il convient de remarquer;

$$ds = -\sqrt{n}\,\frac{d\alpha}{\cos\beta}, \quad ds = -\sqrt{(n+1)}\,\frac{d\beta}{\cos\alpha}:$$

De ce qui precede résulte le mode de génération suivant pour les courbes elliptiques:—*Si le triangle OMP varie de telle manière que le sommet O reste fixe, que les deux côtés OP et MP soient constamment égaux le premier à $\sqrt{n}$, le second à $\sqrt{(n+1)}$, et que de plus le déplacement, infiniment petit MM′ du point M ait lieu à chaque instant suivant la droite qui joint ce point au centre du cercle circonscrit, au triangle générateur, le point M engendrera la courbe elliptique que correspond au nombre n.*

On a ainsi en particulier la démonstration des théorèmes II. et III. du paragraphe II.; lesquels sont relatifs seulement à la lemniscate.

On obtient aisément l'expression du rayon de courbure; soit ε l'angle que fait la normal avec l'axe polaire, on aura

$$\varepsilon = \theta - (\alpha + \beta)$$

car $(\alpha + \beta)$ est l'angle de la normal avec le rayon vecteur; on a, en différentiant, l'angle de contingence $d\varepsilon$

$$d\varepsilon = d\theta - d\alpha - d\beta = \frac{3r^2 - (2n+1)}{\Delta}\,\frac{dr}{r},$$

et pour le rayon de courbure

$$\frac{ds}{d\varepsilon} = R = \frac{2r\sqrt{\{n(n+1)\}}}{3r^2 - (2n+1)},$$

§ IV. Les courbes dont je viens de parler sont celles que j'ai désignées sous le nom de courbes elliptiques de la première classe, dans une note insérée au Journal de M. Liouville (t. x. 1846); on voit qu'il s'en trouve une dont l'arc sera identique à telle fonction elliptique de première espèce, que l'on voudra. Les courbes de la troisième classe sont définies par l'équation

$$x + y\sqrt{(-1)} = ce^{\omega\sqrt{-1}}\int\frac{(z-a)^m\,(z+a)^n}{(z-\alpha)^{m+1}\,(z+\alpha)^{n+1}}\,dz$$

où la quantité $\dfrac{(a+\alpha)^2}{4a\alpha} = \zeta$, est une racine de l'équation

$$\frac{d^n\zeta^m(1-\zeta)^n}{d\zeta^n} = 0.$$

Si l'on a $n = m =$ un nombre entier impair $2\mu + 1$, cette équation aura toujours pour racine $\tfrac{1}{2}$, en sorte que toutes les classes de rang impair comprendront une courbe dont l'arc sera identique à l'arc de lemniscate. On a dans ce cas $a^2 = -\alpha^2 = \sqrt{(-1)}$, et ces nouvelles courbes sont définies par l'équation

$$x + y\sqrt{(-1)} = ce^{\omega\sqrt{(-1)}}\int\frac{\{z^2 - \sqrt{(-1)}\}^{2\mu+1}}{(z^2 + \sqrt{-1})^{2\mu+2}}\,dz,$$

the given constants, are necessarily real. By means of these assumptions, and by dividing the first of equations (8) by mn, the second by nl, the third by lm, we reduce them to the two following:

$$\frac{f(fl+gm+hn)+\alpha l}{l}=\frac{g(fl+gm+hn)+\beta m}{m}$$

$$=\frac{h(fl+gm+hn)+\gamma n}{n} \quad \ldots\ldots(15).$$

If we put $$S=fl+gm+hn \quad \ldots\ldots\ldots\ldots\ldots (16),$$
and denote each member of equations (15) by K, we find

$$l=\frac{Sf}{K-\alpha},\quad m=\frac{Sg}{K-\beta},\quad n=\frac{Sh}{K-\gamma} \quad \ldots\ldots(17);$$

and therefore (except in the case of $S=0$, when some of the principal axes are indeterminate), we have, by (16),

$$\frac{f^2}{K-\alpha}+\frac{g^2}{K-\beta}+\frac{h^2}{K-\gamma}=1 \quad \ldots\ldots\ldots(18).$$

This equation determines three real values for K, (see First Series, vol. IV. p. 229), one of which lies between γ and β, another between β and α, and the third between α and ∞ * (α, β, γ being supposed to be in order of descending magnitude). Any one of these values, substituted for K in (15) give values of the ratios $l:m:n$, which fix the position of a principal axis. Hence there are three, and only three, principal axes through any point. These may be shewn to form a rectangular system, in the following manner, which is quite similar to a method given in the *Mathematical Journal* (First Series, vol. III. p. 291), for demonstrating the perpendicularity of two lines in space, in a corresponding case.

Let K_1, K_2 be two roots of the cubic (18), and (l_1, m_1, n_1) (l_2, m_2, n_2) the corresponding principal axes. Substituting for K the values K_1 and K_2 successively, in (18), we obtain, by subtraction,

$$(K_1-K_2)\left\{\frac{f^2}{(K_1-\alpha)(K_2-\alpha)}+\frac{g^2}{(K_1-\beta)(K_2-\beta)}+\frac{h^2}{(K_1-\gamma)(K_2-\gamma)}\right\}$$
$$=0.$$

Unless K_1 be equal to K_2 the second factor must vanish, and therefore, by (17),

$$l_1l_2+m_1m_2+n_1n_2=0;$$

which shews that the principal axes corresponding to any two different roots of the cubic are at right angles. Hence if the

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* If one of the quantities f, g, or h vanishes, there will be a root equal to the corresponding quantity α, β, or γ.

[redacted]

Since all the quantities $\alpha, \beta, \gamma, f, g, h$ are, in the actual problem, whether of the reduction of the general equation of the second degree, or of the determination of principal axes of a solid, necessarily real, each of the seven squares must vanish, if the sum vanishes. Hence we obtain seven equations as the conditions for the equality of two of the roots of the cubic, which, the special cases of any of the quantities f, g, h vanishing being excluded, are equivalent to the two distinct equations

$$\alpha = \beta = \gamma.$$

If imaginary values of the coefficients were admissible, one condition would of course be sufficient.

15. In the first part of this paper (§ 10) it was shewn that the condition for there being a principal axis in the plane of (xy) is

$$B'(FA' - B'C') - A'(GB' - C'A') = 0,$$

(which is the same as (a) § 10, since $F - G = B - A$). Each member of this equation may be divided by $A'B'$, unless either A' or B' vanishes, in which case, to satisfy the equation, another also of the three quantities A', B', C' must vanish, and one of the axes of coordinates is a principal axis. Hence, if none of the axes of coordinates be a principal axis, the condition that there may be a principal axis in the plane of (xy) is

$$F - \frac{B'C'}{A'} = G - \frac{C'A'}{B'} \; \dots\dots\dots\dots (a).$$

Similarly the condition that there may be a principal axis in the plane of yz, is

$$G - \frac{C'A'}{B'} = H - \frac{A'B'}{C'} \; \dots\dots\dots\dots (b).$$

If there be a principal axis in each of the planes (xy) and (yz), equations (a) and (b) will hold simultaneously, and therefore the conditions are

$$F - \frac{B'C'}{A'} = G - \frac{C'A'}{B'} = H - \frac{A'B'}{C'} \; \dots\dots (c).$$

From the symmetry of the three members of these equations we infer that, if there be a principal axis in the plane (xy), and another in (yz), there shall also be a principal axis in the plane of (zx), and the conditions for this case are the same as those for two of the roots of the cubic being equal.

These results might also have been arrived at by the following simple considerations. It is impossible to find, in the planes of (xy) and (yz), two lines at right angles, of which neither is one of the axes of coordinates. Hence,

[redacted]

$$\begin{cases} (F-K)(G-K)(H-K)-A'^2(F-K)-B'^2(G-K)-C'^2(H-K) \\ \qquad\qquad\qquad\qquad\qquad\qquad + 2A'B'C' = 0 \ldots (b)^* \\ \dfrac{f^2}{K-\left(F-\dfrac{B'C'}{A'}\right)}+\dfrac{g^2}{K-\left(G-\dfrac{C'A'}{B'}\right)}+\dfrac{h^2}{K-\left(H-\dfrac{A'B'}{C'}\right)}-1=0 \ldots (b'), \end{cases}$$

the quantities F, G, H, f, g, h, being given by the equations

$$F=\tfrac{1}{2}(B+C-A), \quad G=\tfrac{1}{2}(C+A-B), \quad H=\tfrac{1}{2}(A+B-C),$$

$$f^2=\frac{B'C'}{A'}, \quad g^2=\frac{C'A'}{B'}, \quad h^2=\frac{A'B'}{C'}.$$

It may be algebraically verified that (a) and (a') are identical, as also (b) and (b'); and that (b) or (b') may be deduced from (a) or (a') by assuming

$$P=F+G+H-K \ldots\ldots\ldots\ldots (c).$$

The roots of (a) or (a') are the three principal moments of inertia, and the roots of (b) or (b') substituted in (c), give the same quantities.

18. I shall conclude this paper by applying some of the formulæ given above to the solution of the following problem.

"Having given the moments of inertia of a body round the principal axes through its centre of gravity, shew how to determine the position of the principal axes through any other point, and the moments of inertia round them."—*St. Peter's College Examination Papers*, May 1845.

Let O be the centre of gravity; OX, OY, OZ principal axes; A, B, C the moments of inertia of the body round them: and, according to our previous notation, let

$$F=\Sigma\delta\mu x^2, \quad G=\Sigma\delta\mu y^2, \quad H=\Sigma\delta\mu z^2,$$

so that $\quad A=G+H, \quad B=H+F, \quad C=F+G \ldots (a).$

Let P be any point (ξ, η, ζ) for which it is required to determine the principal axes and moments. The integrals (or sums) which will enter as coefficients in the equations for determining the required quantities will be

$$\Sigma\delta\mu(x-\xi)^2, \quad \Sigma\delta\mu(y-\eta)^2, \quad \Sigma\delta\mu(z-\zeta)^2,$$

$$\Sigma\delta\mu(y-\eta)(z-\zeta), \quad \Sigma\delta\mu(z-\zeta)(x-\xi), \quad \Sigma\delta\mu(x-\xi)(y-\eta).$$

Expanding these expressions, and taking into account the properties of the axes of coordinates (principal axes through the centre of gravity), we find that they are respectively equal to

$$F+\mu\xi^2, \quad G+\mu\eta^2, \quad H+\mu\zeta^2,$$

$$\mu\eta\zeta, \qquad \mu\zeta\xi, \qquad \mu\xi\eta,$$

* This is the form of the "discriminating cubic" usually given. (See First Series, vol. I. p. 35; or *Earnshaw's Dynamics*, Art. 190.)

with the surface (f), which we shall, for distinction, call the *central ellipsoid* of the body.*

The principal axes through any point of a solid body are normals to the three surfaces of the second order confocal with the central ellipsoid, which intersect in that point.†

For determining the principal moments corresponding to the point $(\xi\,\eta\,\zeta)$, since $F+G+H+\mu(\xi^2+\eta^2+\zeta^2)$ is the sum of the roots of the cubic (b), we have

$$P_1 = F+G+H+\mu(\xi^2+\eta^2+\zeta^2)-K_1,$$

where P_1 is the moment of inertia round the axis given by the root K_1 of the cubic. Hence if we take

$$K = F+G+H+\mu(\xi^2+\eta^2+\zeta^2)-P \ldots\ldots (g)$$

in (d), the equation thus obtained,

$$\frac{\xi^2}{\xi^2+\eta^2+\zeta^2+a^2-\dfrac{P}{\mu}}+\frac{\eta^2}{\xi^2+\eta^2+\zeta^2+b^2-\dfrac{P}{\mu}}+\frac{\zeta^2}{\xi^2+\eta^2+\zeta^2+c^2-\dfrac{P}{\mu}}=1 \quad (h),$$

determines three real values for P, which are the moments of inertia round principal axes through $(\xi\,\eta\,\zeta)$.

Glasgow, Jan. 6, 1846.

POSTSCRIPT.

20. If in equation (h) of the last section, P have a given value, Π; and if x, y, z be any values of ξ, η, ζ which satisfy the equation in this case, the locus of $x\,y\,z$ will be a surface

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* In the first part of this paper (see p. 130, § 6 and Note) two ellipsoids were mentioned, either of which, different for different points of the body, may be described round any point as centre, and affords a geometrical construction for determining the principal axes of the body through this point. The *central ellipsoid* is unique in a solid body; its principal axes coincide in direction with the principal axes of the body through the centre of gravity; and the semiaxes are equal to $\sqrt{\dfrac{A}{\mu}}, \sqrt{\dfrac{B}{\mu}}, \sqrt{\dfrac{C}{\mu}}$ (radii of gyration), and thus its form, position, and magnitude, are entirely fixed in the body. The axes of either of the other ellipsoids for any point of the body coincide with the principal axes of the body through the point, and, arbitrary in absolute magnitude, are proportional to $\dfrac{1}{\sqrt{A}}, \dfrac{1}{\sqrt{B}}, \dfrac{1}{\sqrt{C}}$, or to $\dfrac{1}{\sqrt{F}}, \dfrac{1}{\sqrt{G}}, \dfrac{1}{\sqrt{H}}$, the first being Poinsot's *momental ellipsoid*, and the second the ordinary *ellipsoid of construction*.

† This theorem has also been demonstrated by Mr. Townsend (Fellow of Trinity College, Dublin). His investigation, which is connected with the geometrical properties of confocal surfaces, and of enveloping cones, is entirely different from that given above, and will be published in an early Number of the *Journal*. I am informed by Mr. Townsend that another demonstration of the same theorem, which will also be contained in his paper, has been given (but not, so far as I am aware, published), several years ago, by Professor Maccullagh.

from which we deduce, by ordinary processes,

$$\frac{lx}{r^2-\alpha}+\frac{my}{r^2-\beta}+\frac{nz}{r^2-\gamma}=0 \;..........(c),$$

and each member of equations (b) is

$$=\frac{v}{1-Sr^2}, \quad \text{or} = \frac{1}{-Sv}.$$

Hence $\qquad S(r^2-v^2)=1 \;..............(d),$

and we have

$$\left.\begin{aligned} lv(r^2-\alpha)&=x(v^2-\alpha)\\ mv(r^2-\beta)&=y(v^2-\beta)\\ nv(r^2-\gamma)&=z(v^2-\gamma)\end{aligned}\right\} \;........ (e).^*$$

Now lv, mv, nv are the coordinates of Q, and hence if λ, μ, ν be the direction-cosines of PQ, we have

$$\left.\begin{aligned} &\frac{\lambda}{r^2-\alpha}=\frac{\mu}{r^2-\beta}=\frac{\nu}{r^2-\gamma}\\ \text{or} \quad &\frac{\lambda}{r^2+a^2-\frac{\Pi}{\mu}}=\frac{\mu}{r^2+b^2-\frac{\Pi}{\mu}}=\frac{\nu}{r^2+c^2-\frac{\Pi}{\mu}}\end{aligned}\right\} \;......(f),$$

which prove the theorem enunciated, since, by (b), § 18, these are the equations for determining the principal axis, corresponding to a root Π of the cubic equation which determines the three principal moments at any point ($x\,y\,z$).

The locus of points on any surface

$$\frac{x^2}{a^2+\Theta}+\frac{y^2}{b^2+\Theta}+\frac{z^2}{c^2+\Theta}=1 \;........(g),$$

(confocal with the central ellipsoid) for which one of the principal moments is equal to Π, is determined by the equations (a) and (g) considered as simultaneous. By subtracting the former from the latter, we get

$$\left(r^2-\frac{\Pi}{\mu}-\Theta\right)\left\{\frac{x^2}{(a^2+\Theta)\left(r^2+a^2-\frac{\Pi}{\mu}\right)}+\frac{y^2}{(b^2+\Theta)\left(r^2+b^2-\frac{\Pi}{\mu}\right)}+\frac{z^2}{(c^2+\Theta)\left(r^2+c^2-\frac{\Pi}{\mu}\right)}\right\}=0.$$

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* See *Math. Journal*, vol. I. (First Series) p. 8, or Gregory's *Examples*, p. 232, where the same formulæ occur in the investigation of the wave surface.

22 As an illustration of what has been proved in the last section, let us suppose $\Theta = 0$, so that (g) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the equation of the central ellipsoid. It may be readily shewn that the moments of inertia round normals to this surface lie between the limits c^2 and a^2 (the mass being taken as unity), and those round lines touching it, between $b^2 + c^2$ and $a^2 + b^2$. Hence if Π have all values from c^2 to a^2 given it in succession, all the equimomental surfaces which intersect the central ellipsoid in a real spherical conic will be obtained; and if all values from $b^2 + c^2$ to $a^2 + b^2$ be given, all the equimomental surfaces which intersect the central ellipsoid in real lines of curvature will be obtained. The values of Π from $b^2 + c^2$ to $c^2 + a^2$ will give equimomental surfaces which cut the ellipsoid in the lines of curvature corresponding to the confocal hyperboloids of one sheet; and the other set of lines of curvature are therefore given by values of Π between $c^2 + a^2$ and $a^2 + b^2$.

Since $b^2 + c^2 > a^2$, except in the limiting case of the body being reduced to a portion of the plane of yz, when these quantities are equal, the values of Π which make one curve of intersection real must make the other imaginary. Hence the same equimomental surface can only cut the central ellipsoid in a "spherical conic" or a line of curvature, but not in both. The same may be shewn to be true for any confocal ellipsoid (but not for any of the hyperboloids). Hence any of the equimomental surfaces which cuts one of the ellipsoids must do so along either a spherical conic or a line of curvature, but not along both.

There are many interesting subjects of investigation relative to these equimomental surfaces which present themselves; such as the properties of the three equimomental surfaces which intersect in a given point and of the curves along which they cut one another, the forms of the different classes of equimomental surfaces, and the remarkable properties of principal axes at different points (corresponding to the *conical refraction* of light), which are due to the singular points of these surfaces. These may be discussed in a future communication; but the length to which this paper has already been extended, prevents me from going farther in the subject at present.

St. Peter's College, May 8, 1846.

[redacted]

are equivalent to two independent equations, *i.e.* the third can be deduced from the two first. Now the first equation is that of an ellipsoid (or generally a surface of the second order, since a, b, c are not necessarily real). The second is that of what may be called a conjugate equimomental surface, defining the term as follows: "The conjugate equimomental surfaces of an ellipsoid (or surface of the second order) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, are the equimomental surfaces derived in the usual manner from any surface of the second order $\frac{x^2}{h-a^2}+\frac{y^2}{h-b^2}+\frac{z^2}{h-c^2}=1$, which is confocal with the conjugate surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$ of the given ellipsoid," viz. by measuring along any line through the centre distances equal to the axes of the section by a plane through the centre perpendicular to this line, and taking the locus of the points so determined for the equimomental surface. The third equation is that of a surface confocal with the given ellipsoid; hence the theorem, "The curves of curvature of a given ellipsoid lie upon a system of conjugate equimomental surfaces."

But since the first and second equations are evidently satisfied by the combination of the first equation with the relation $r^2=h$, which is that of a sphere, we have also, "The curve of intersection of the ellipsoid with any one of the conjugate equimomental surfaces, is composed of the line of curvature, and of a spherical conic." And these two curves being each of them of the fourth order make up the complete curve of intersection, which should obviously be of the eighth order.

It would be an interesting question to determine the relations existing between the curve of curvature and the spherical conic, which have been thus brought into connection by means of the conjugate equimomental surfaces; *i.e.* between the two curves obtained by combining the equation of the ellipsoid with

$$\frac{x^2}{a^2+k}+\frac{y^2}{b^2+k}+\frac{z^2}{c^2+k}=1,$$

$$r^2=a^2+b^2+c^2+k,$$

respectively: but it will be sufficient at present to have suggested the problem.

solved parts of that resultant in any directions be equal to nothing; then will the forces either equilibrate or compound a single resultant passing through that point.

If, therefore, OZ be a *permanent* axis of rotation (the point O of the body alone being fixed), then must the quantities $\Sigma xzdm$ and $\Sigma yzdm$ be $= 0$ for *every* pair of coordinate axes through O at right angles to OZ, and conversely. If these analytical quantities be both $= 0$, then for the point O will the resultant moment of the centrifugal forces vanish, or the axis be permanent.

2. The forces will not equilibrate except in the particular case when the point is the centre of gravity; for, the components of their resultant along OX and OY are respectively $\omega^2\Sigma xdm$ and $\omega^2\Sigma ydm = \omega^2\bar{x}M$ and $\omega^2\bar{y}M$.

Hence, an axis which is principal for one point of a body is principal for no other point on itself, and therefore for no other point whatever, except in the particular case of the centre of gravity whose principal axes are principal for all points along them. This is evident, for when any number of forces compound a single resultant, their resultant moment with respect to an assumed point will vanish only when that point is on the resultant, except in the particular case when that resultant is nothing, that is, when the forces are in equilibrium.

Again, since in general systems of forces neither equilibrate nor compound a single resultant, it follows that an axis taken at random in a body may not be principal for any point at all; and that such is often actually the case, appears at once from the values for the components of the resultant force and moment, which for *every* point on the axis must be such that $\Sigma xzdm : \Sigma yzdm :: \Sigma xdm : \Sigma ydm$, in order that the resultants of the two systems of parallel forces, ω^2xdm in the plane of xz and ω^2ydm in the plane of yz, which both pass through the axis OZ, should meet it at the same point, and therefore compound a single resultant: but this proportion does not in general hold.

Hence, though the number of principal axes in a body may perhaps be infinite, it may still be small in comparison with the number of axes which are not principal.

3. If on every line diverging from any point O of a solid body we measure off a portion r, such that the square of its reciprocal multiplied by the mass of the body shall equal the moment of inertia round it; then, as is well known, will the locus of the extremity of r be an ellipsoid (the "Momental" Ellipsoid of Poinsot). This is usually shewn as follows.

For each point O, this ellipsoid is, from its very nature, fixed in magnitude and position with respect to the body, and is of course independent of the arbitrarily assumed directions of the coordinate axes; though if the latter be changed, the coefficients $A'\ B'\ C'\ A''\ B''\ C''$ in the new equation will no longer represent the *same* sums as before, but will be equal to the similar sums with respect to the new axes. Conversely, therefore, this ellipsoid once determined gives us the values of these six integrals for every rectangular system of coordinate axes assumed at pleasure through O.

Now, in *every* ellipsoid (and therefore in the above) there exist *three* lines, passing through its centre and at right angles to each other (and generally but three), to which as axes of coordinates if the surface be referred, the coefficients $A''\ B''\ C''$ will all disappear from the equation, and for each of which, whatever be the directions of the other two coordinate axes, provided they be both at right angles to the first, two of the same quantities will vanish. This, therefore, being the analytical property of a principal axis, proves that for every point of a solid body there are, at least, three principal axes at right angles to each other, the axes, viz., of its momental ellipsoid.*

4. By means of the momental ellipsoid we may also arrive at the same result from the dynamical property of a principal axis.

For, suppose the body to revolve round any diameter $2r$ of that ellipsoid, with which, as the coordinate rectangular axes are quite arbitrary, let one of them, that of z, coincide; then, in the equation of the surface (3),

$$A'x^2 + B'y^2 + C'z^2 - 2A''yz - 2B''zx - 2C''xy = M,$$

will the quantities A'' and B'', multiplied each by the square of the angular velocity, be equal to the components in the planes of zy and of zx respectively of the resultant moment of all the centrifugal forces.

A'' and B'' are, therefore, proportional to the cosines of the angles which the plane of that resultant passing through the axis of z makes with the planes of zy and of zx respectively, and their *signs* are the same as those of the *directions* in which the components tend to draw the axis of rotation.

* This proof is due to Professor MacCullagh, and was given by him at his mathematical lectures in Trinity College, Dublin.

To shew this, let I' I'' be the moments of inertia round the semiaxes r' r'' of the ellipse in which the plane intersects the momental ellipsoid at the centre of gravity, ρ' ρ'' the radii of the same parallel and perpendicular to one of the equimomental axes, d the distance between that axis and ρ', and α β the angles which ρ'' or d makes with r' and r''; then, denoting by I the constant moment common to all the axes, we shall have $\frac{M}{\rho^2} + M.d^2 = I$, from which, since $\frac{1}{\rho^2} = \frac{1}{r'^2}\cos^2\alpha + \frac{1}{r''^2}\sin^2\alpha$, $\frac{M}{\rho^2} = I'\cos^2\alpha + I''\sin^2\alpha$, and we get

$$M.d^2 = (I - I')\cos^2\alpha + (I - I'')\sin^2\alpha \;\ldots\ldots\; (a):$$

the equimomental axes therefore all envelope a central conic, the squares of whose semi-axes are $\frac{I - I'}{M}$ and $\frac{I - I''}{M}$, and which will be therefore an ellipse or hyperbola, as the case may be.

For different systems of equimomental axes in the same plane I will vary but I' and I'', and their axes will remain the same: hence we know that the system of conics enveloped by all the systems of axes will be coaxal and confocal, their common axes being those of the section of the central ellipsoid.

The ellipse a', b', or $\sqrt{\frac{M}{I'}}$, $\sqrt{\frac{M}{I''}}$, with which the conics are all confocal, is obviously the ellipse cyclo-polar reciprocal* to the ellipse section of the momental ellipsoid by the plane of the axes, the radius of the reciprocating circle being 1, and the pole being the centre of that ellipse, or the centre of gravity.

9. Now, if a system of equimomental axes, lying all in any plane whatever, be projected orthographically upon a parallel plane through the centre of gravity (and therefore upon *any* parallel plane), the projections will be also a system of equimomental axes.

For the moment round each axis will exceed that round its projection by $M\delta^2$, δ being the distance between the planes.

---

* The terms *cyclo* and *sphero* polar reciprocal, were first introduced by Mr. Ingram, Fellow of Trinity College, Dublin, in order to distinguish from the general class of polars with respect to *any* curve or surface of the *second* order (which class alone possesses the important property of *reciprocality*) that particular class where the reciprocating curve or surface is a *circle* or *sphere*.

[redacted]

More generally, if at every point of any axis through the centre of gravity perpendiculars be erected all round the axis, and that portions be taken on each whose squares, multiplied each by the mass of the body, shall equal their moments of inertia; then, for the same reason as above, will the locus of the extremities of these portions be a surface all whose sections by planes through the axis will be equilateral hyperbolas.

(It is, perhaps, needless to say that this surface will not be an hyperboloid, for, except in the particular case just noticed, its sections perpendicular to the axis will not be curves of the second order. See Note Art. 3.)

If at all points of any plane drawn at will through the centre of gravity of a body, we erect perpendiculars whose squares multiplied each by the mass shall equal their moments of inertia, the locus of their extremities will be an hyperboloid of two sheets of revolution round the perpendicular through the centre of gravity:

For, drawing through that perpendicular any two rectangular planes, let the coordinates referred thereto of the extremity of one of the variable perpendiculars z be xy, then have we $Mz^2 = Mz_1^{\,2} + M(x^2 + y^2)$, and therefore $x^2 + y^2 - z^2 = z_1^{\,2}$; the meridians therefore are equilateral hyperbolas, and the surface is of revolution round their transverse axis.

(The plane obviously need not pass through the centre of gravity, and the only difference will be that the centre of the hyperboloid will not be that centre, but its projection on the plane.)

For every plane through the centre of gravity we have a different hyperboloid; these are all connected by the property that the squares of their axes represent their moments of inertia. If, therefore, at the vertices of each we draw tangent planes, the whole system of planes thus produced will envelope an ellipsoid (of which we have spoken and shall frequently use again), the ellipsoid of gyration, sphero-polar reciprocal with respect to the sphere $x^2 + y^2 + z^2 = 1$ of the momental ellipsoid at the centre of gravity. This is evident, for the coincident radii of the latter surface are inversely as the axes of the hyperboloids, shewing that the extremities of these radii are the poles with respect to that sphere of the tangent planes in question (Note Art. 3), and that, conversely, the locus of the vertices of the polar reciprocal system of hyperboloids is the momental ellipsoid at the centre of gravity.

angular point of the figure. Then it will be obvious from the figure itself that

$$\text{perimeter} = 4n\rho \tan \frac{\pi}{2m},$$

$$HK = 2\rho \sec \frac{\pi}{2m} \sin \frac{n\pi}{m},$$

$$AL = \frac{\rho \sin \dfrac{n\pi}{m}}{2n \sin \dfrac{\pi}{2m}},$$

and
$$LM^2 = \rho^2 \left\{1 - \frac{\sin^2 \dfrac{n\pi}{m}}{4n^2 \sin^2 \dfrac{\pi}{2m}}\right\} \dots\dots\dots\dots (1).$$

Now the coordinates of the points of contact referred to the centre A and axis AF will be

$$\left(\rho, \frac{\pi}{2m}\right), \left(\rho, \frac{3\pi}{2m}\right) \dots \left\{\rho, \frac{(2n-1)\pi}{2m}\right\};$$

and if $r\theta$ denote the arbitrary point N, and the expressions for the several squares enunciated be formed, we shall have

$$NL^2 = r^2 - 2r.AL \cos\theta + AL^2$$

$$= r^2 - 2r\rho . \frac{\sin \dfrac{n\pi}{m} \cos\theta}{2n \sin \dfrac{\pi}{2m}} + \rho^2 . \frac{\sin^2 \dfrac{n\pi}{m}}{4n^2 \sin^2 \dfrac{\pi}{2m}} \dots (2),$$

and from (1, 2), we get at once

$$NL^2 + LM^2 = r^2 - r\rho \frac{\sin \dfrac{n\pi}{m}}{n \sin \dfrac{\pi}{2m}} \cos\theta + \rho^2 \dots (3).$$

Also taking the squares of the lines drawn from N to the points of contact equidistant from the circular origin in pairs, we shall have them represented by

$$\left.\begin{array}{l} r^2 - 2r\rho \cos\left(\theta - \dfrac{\pi}{2m}\right) + \rho^2 \\ r^2 - 2r\rho \cos\left(\theta + \dfrac{\pi}{2m}\right) + \rho^2 \end{array}\right\},$$

$$\left.\begin{array}{l} r^2 - 2r\rho\cos\left(\theta - \dfrac{3\pi}{2m}\right) + \rho^2 \\ r^2 - 2r\rho\cos\left(\theta + \dfrac{3\pi}{2m}\right) + \rho^2 \end{array}\right\},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left.\begin{array}{l} r^2 - 2r\rho\cos\left\{\theta - \dfrac{(2n-1)\pi}{m}\right\} + \rho^2 \\ r^2 - 2r\rho\cos\left\{\theta + \dfrac{(2n-1)\pi}{m}\right\} + \rho^2 \end{array}\right\}.$$

Expanding the cosines and adding, and putting S^2 for the sum of the squares of the lines, we have

$$S^2 = 2n\,(\rho^2 + \gamma^2) - 4r\rho\cos\theta\left(\cos\frac{\pi}{2m} + \cos\frac{3\pi}{2m} + \ldots\right).$$

But $\cos\dfrac{\pi}{2m} + \cos\dfrac{3\pi}{2m} + \ldots + \cos\dfrac{(2n-1)\pi}{2m} = \dfrac{\sin\dfrac{n\pi}{m}}{2\sin\dfrac{\pi}{2m}}$,

and hence $\quad S^2 = 2\,(\rho^2 + r^2) - 2r\rho\,\dfrac{\sin\dfrac{n\pi}{m}}{\sin\dfrac{\pi}{2m}}\cos\theta \ldots\ldots (4).$

Wherefore, multiplying (3) by $2n$, we have (4), which shews that in this case the proposition is true; viz. that

$$S^2 = 2n\,(NL^2 + LM^2).$$

2. *Let the polygon have* $2n + 1$ *sides.*

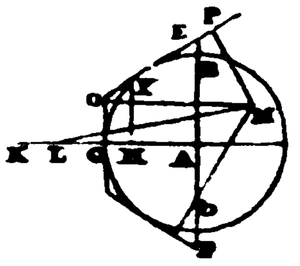
Take origin and axis as before, the axis now passing through one of the points of contact of the figure. Then we shall have, the segment being denoted by $\dfrac{(2n+1)\pi}{m}$,

$$\text{perimeter} = 2\,(2n+1)\,\rho\tan\frac{\pi}{2m},$$

$$HK = 2\rho\sec\frac{\pi}{2m}\sin\frac{(2n+1)\pi}{2m},$$

$$AL = \frac{\rho\sin\dfrac{(2n+1)\pi}{2m}}{(2n+1)\sin\dfrac{\pi}{2m}},$$

PROP. VIII. GENERAL THEOREMS.



"*Let there be any circle whose centre is* A, *and let* BCD *be a semicircle, and* BD *the diameter of the circle; about the semicircle let there be any regular figure described, and let the sides of the figure next to* BD *meet* BD *in* E, F; *bisect the semicircle in* G *and join* AG; *and in* AG *take the point* H *on the same side of the centre* A *with the point* G, *and let* AG *be to* AH *as the sum of the sides of the figure to* EF; *and let the rectangle* HAK *be equal to the square of the semidiameter, and* HL *be equal to* AH: *if from any point* M *there be drawn* MN, MO, MP, *etc., perpendicular to the sides of the figure circumscribed about the semicircle, and likewise there be drawn* ML *to the point* L; *twice the sum of the squares of the perpendiculars* MN, MO, MP, *etc., will be equal to the multiple of the square of* ML *by the number of the sides of the figure together with the multiple of the rectangle* KLA *by the same number.*"

1. *Let the figure have* 2*n sides.*

Taking the line AG as angular origin, the angular axis will pass through an angular point of the circumscribed figure. Denote the radius by ρ: then the polar equations of the sides of the figure whose points of contact lie respectively above and below the circular origin will be (Hutton's *Course*, vol. II. p. 264, *twelfth edition*),

$$\rho = r\cos\left(\theta - \frac{\pi}{4n}\right),\quad \rho = r\cos\left(\theta - \frac{3\pi}{4n}\right),\ldots$$

$$\rho = r\cos\left\{\theta - \frac{(2n-1)\pi}{4n}\right\},$$

$$\rho = r\cos\left(\theta + \frac{\pi}{4n}\right),\quad \rho = r\cos\left(\theta + \frac{3\pi}{4n}\right),\ldots$$

$$\rho = r\cos\left\{\theta + \frac{(2n-1)\pi}{4n}\right\},$$

and the perpendiculars from the arbitrary point $r\theta\,(M)$ upon these will be (*ib.*)

$$\rho - r\cos\left(\theta - \frac{\pi}{4n}\right),\quad \rho - r\cos\left(\theta - \frac{3\pi}{4n}\right),\ldots$$

$$\rho - r\cos\left\{\theta - \frac{(2n-1)\pi}{4n}\right\},$$

[redacted]

$$\rho - r\cos\left(\theta + \frac{\pi}{4n}\right), \quad \rho - r\cos\left(\theta + \frac{3\pi}{4n}\right), \ldots$$

$$\rho - r\cos\left\{\theta + \frac{(2n-1)\pi}{4n}\right\}.$$

Taking the sums of the squares of these in pairs as they stand beneath each other, and expressing the results in multiple cosines, we get

$$2\rho^2 + r^2 - 4r\rho\cos\theta\cos\frac{\pi}{4n} + r^2\cos 2\theta\cos\frac{\pi}{n},$$

$$2\rho^2 + r^2 - 4r\rho\cos\theta\cos\frac{3\pi}{4n} + r^2\cos 2\theta\cos\frac{3\pi}{n},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$2\rho^2 + r^2 - 4r\rho\cos\theta\cos\frac{(2n-1)\pi}{4n} + r^2\cos 2\theta\cos\frac{(2n-1)\pi}{n}.$$

Now the sum of these being taken, the column in $\cos 2\theta$ vanishes, its coefficient being $\dfrac{\sin\pi\cos\pi}{\sin\frac{\pi}{n}} = 0$; and the column in $\cos\theta$ has for its coefficient the value $\dfrac{\sin\frac{\pi}{4}\cos\frac{\pi}{4}}{\sin\frac{\pi}{4n}} = \dfrac{1}{2\sin\frac{\pi}{4n}}$:

and hence twice the sum of the squares, $2S^2$, of the $2n$ perpendiculars is

$$2S^2 = 4n\rho^2 + 2nr^2 - 4r\rho\frac{\cos\theta}{\sin\frac{\pi}{4n}} \quad \ldots\ldots\ldots (1).$$

Again, from the obvious properties of the figure,

$$\text{perimeter} = 4n\rho\tan\frac{\pi}{4n},$$

$$AH = \frac{\rho}{2n\sin\frac{\pi}{4n}},$$

$$AK = 2n\rho\sin\frac{\pi}{4n},$$

$$AL = \frac{\rho}{n\sin\frac{\pi}{4n}},$$

[redacted]

$$KL = \rho \, . \frac{2n^2 \sin^2 \dfrac{\pi}{4n} - 1}{n \sin \dfrac{\pi}{4n}},$$

$$KL.LA = 2\rho^2 - \frac{\rho^2}{n^2 \sin^2 \dfrac{\pi}{4n}},$$

$$ML^2 = LA^2 - 2LA.AM \cos LAM + AM^2$$

$$= \frac{\rho^2}{n^2 \sin^2 \dfrac{\pi}{4n}} - \frac{2r\rho \cos \theta}{n \sin \dfrac{\pi}{4n}} + r^2;$$

and hence $ML^2 + KL.LA = 2\rho^2 + r^2 - \dfrac{2r\rho \cos \theta}{n \sin \dfrac{\pi}{4n}}$ .... (2).

Whence multiplying this by $2n$, we have the same result as in (1); and this identity of value is that enunciated in the proposition, viz.

$$2S^2 = 2n\,(ML^2 + KL.LA).$$

2. *Let the figure have* $2n + 1$ *sides.*

In this case the angular axis will pass through the middle point of contact; and the perpendiculars from $r\theta$ upon the sides of the figure, whose points of contact are respectively above and below the circular origin, will be (omitting that upon the side whose point of contact is the circular origin)

$$\rho - r \cos\left(\theta - \frac{\pi}{2n+1}\right), \quad \rho - r \cos\left(\theta - \frac{2\pi}{2n+1}\right) \ldots.$$

$$\rho - r \cos\left(\theta - \frac{n\pi}{2n+1}\right),$$

$$\rho - r \cos\left(\theta + \frac{\pi}{2n+1}\right), \quad \rho - r \cos\left(\theta + \frac{2\pi}{2n+1}\right) \ldots.$$

$$\rho - r \cos\left(\theta + \frac{n\pi}{2n+1}\right).$$

Twice the sum of the squares of these being taken, and the results expressed in multiple cosines, we get

$$2\left\{2\rho^2 + r^2 - 4r\rho \cos \theta \cos \frac{\pi}{2n+1} + r^2 \cos 2\theta \cos \frac{2\pi}{2n+1}\right\},$$

$$2\left\{2\rho^2 + r^2 - 4r\rho\cos\theta\cos\frac{2\pi}{2n+1} + r^2\cos 2\theta\cos\frac{4\pi}{2n+1}\right\},$$

. . . .

$$2\left\{2\rho^2 + r^2 - 4r\rho\cos\theta\cos\frac{n\pi}{2n+1} + r^2\cos 2\theta\cos\frac{2n\pi}{2n+1}\right\},$$

the sum of which is

$$2n\,(2\rho^2 + r^2) - 4r\rho\cos\theta\left\{\frac{1}{\sin\frac{\pi}{2n+1}} - 1\right\} - r^2\cos 2\theta \,..(1).$$

Also double the square of the omitted perpendicular is

$$2\rho^2 + r^2 - 4r\rho\cos\theta + r^2\cos 2\theta,$$

which, added to (1), gives

$$2S^2 = (2n+1)(2\rho^2 + r^2) - 4r\rho.\frac{\cos\theta}{\sin\frac{\pi}{2n+1}} \,\ldots\ldots(2).$$

Again, we have, as in the preceding case,

$$\text{perimeter} = 2\,(2n+1)\,\rho\tan\frac{\pi}{2\,(2n+1)},$$

$$AH = \frac{\rho}{(2n+1)\sin\frac{\pi}{2\,(2n+1)}},$$

$$AK = (2n+1)\,\rho\sin\frac{\pi}{2\,(2n+1)},$$

$$AL = \frac{\pi}{(2n+1)\sin\frac{2\rho}{2\,(2n+1)}},$$

$$KL = \rho.\frac{(2n+1)^2\sin^2\frac{\pi}{2\,(2n+1)} - 2}{(2n+1)\sin\frac{\pi}{2\,(2n+1)}},$$

$$KL.LA = 2\rho^2 - \frac{4\rho^2}{(2n+1)^2\sin^2\frac{\pi}{2\,(2n+1)}},$$

$$LM^2 = \frac{4\rho^2}{(2n+1)^2\sin^2\frac{\pi}{2\,(2n+1)}} - \frac{4r\rho\cos\theta}{(2n+1)\sin\frac{\pi}{2\,(2n+1)}} + r^2;$$

whence, adding, we get

$$KL.LA + LM^2 = 2\rho^2 + r^2 - \frac{4r\rho\cos\theta}{(2n+1)\sin\frac{\pi}{2(2n+1)}},$$

$$\text{or } (2n+1)(KL\,LA + LM^2) = (2n+1)(2\rho^2+r^2) - \frac{4r\rho\cos\theta}{\sin\frac{\pi}{2(2n+1)}} \ldots (3).$$

The identity of (1) and (3) proves the truth of the theorem, also, when the number of sides is odd.

Royal Military Academy, Woolwich, March 25, 1846.

ON ARBOGAST'S FORMULÆ OF EXPANSION.

By Augustus De Morgan,
Professor of Mathematics in University College, London.

§ 1. *General Theory of Derivatives.*

The theory of Arbogast has received so little attention in this country, that no excuse is necessary for an attempt to exhibit its rules in a short and comparatively easy manner. Arbogast himself was more occupied in proving the ease with which his method could be applied to very complicated cases, than in illustrating the connexion of its principles with those of other parts of analysis.

The first attempt of which I know, to write on this subject in English, is contained in the posthumous work* of Mr. West, which is a very complete attempt, as far as series of one variable are concerned. The next is that which I made myself in my work on the Differential Calculus, (at which time I did not know of Mr. West's work): and I am not aware of any other. I think that many mathematicians are under the impression that Argobast's method belongs to the *combinatorial analysis* of Hindenburg and his followers. This, however, any one who carefully examines both will find is not the case.

When Arbogast developes $\phi(a + bx + cx^2 + \ldots)$, he presumes

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* 'Mathematical Treatises, containing, 1. the Theory of Analytical Functions .... By the Rev. John West .... edited .... from his MSS. .... by the late Sir John Leslie ....' Edinburgh, 1838. 8vo.—Mr. West died in Jamaica in 1817, aged 61.

[redacted]

$$
\begin{array}{cccccccc}
a_{-,,,}^{\prime\prime\prime} & & c_{-,,}^{\mathrm{iv}} & & f_{-,}^{\mathrm{v}} & & h^{\mathrm{vi}} & \\
 & b_{-,,}^{\prime\prime\prime} & & e_{-,}^{\mathrm{iv}} & & g^{\mathrm{v}} & & \\
a_{-,,}^{\prime\prime} & & c_{-,}^{\prime\prime\prime} & & f^{\mathrm{iv}} & & h_{,}^{\mathrm{v}} & \\
 & b_{-,}^{\prime\prime} & & e^{\prime\prime\prime} & & g_{,}^{\mathrm{iv}} & & \\
a_{-,}^{\prime} & & c^{\prime\prime} & & f_{,}^{\prime\prime\prime} & & h_{,,}^{\mathrm{iv}} & \\
 & b^{\prime} & & e_{,}^{\prime\prime} & & g_{,,}^{\prime\prime\prime} & & \\
a & & c_{,}^{\prime} & & f_{,,}^{\prime\prime} & & h_{,,,}^{\prime\prime\prime} & \&c. \\
 & b_{,} & & e_{,,}^{\prime} & & g_{,,,}^{\prime\prime} & & \\
a_{,}^{-\prime} & & c_{,,} & & f_{,,,}^{\prime} & & h_{\mathrm{iv}}^{\prime\prime} & \\
 & b_{,,}^{-\prime} & & e_{,,,} & & g_{\mathrm{iv}}^{\prime} & & \\
a_{,,}^{-\prime\prime} & & c_{,,,}^{-\prime} & & f_{\mathrm{iv}} & & h_{\mathrm{v}}^{\prime} & \\
 & b_{,,,}^{-\prime\prime} & & e_{\mathrm{iv}}^{-\prime} & & g^{\mathrm{v}} & & \\
a_{,,,}^{-\prime\prime\prime} & & c_{\mathrm{iv}}^{-\prime\prime} & & f_{\mathrm{v}}^{-\prime} & & h_{\mathrm{vi}} &
\end{array}
$$

with the understanding that all the negatively accented letters are made to vanish at the end of the process, we may then declare the four modes of derivation to be entirely convertible.

Leaving these extensions, however, we proceed to the details of development of $\phi(a + \&c.)$ by means of D_n derivations

and $D_{y:x}$ ones performed upon them. The coefficient of $x^m y^n$ is $D^n_{y:x} D_x^{m+n} \phi a$, or $D_{:}$ is to be performed n times upon

$$\phi' a \,.\, D^{m+n-1} b' + \frac{\phi'' a}{2} D^{m+n-2} b'^2 + \ldots + \frac{\phi^{(m+n)} a}{2 \ldots m+n} b^2 b'^{m+n}.$$

Now, because all the $D_{:}$ derivatives of a vanish, the process is wholly inoperative upon $\phi' a$, $\phi'' a$, &c.; and it is clear that by the difference of dimensions (derivation never producing change of dimension) no term of $D_{:}^p D_x^q b'^r$ can ever be identical with any sum of $D_{:}^s D_x^t b'^u$ if r and u be different. Hence the coefficient of $x^m y^n$ is

$$\phi' a D_{:}^n D_x^{m+n-1} b' + \frac{\phi'' a}{2} D_{:}^n D_x^{m+n-2} b'^2 + \ldots + \frac{\phi^{(m+n)}}{2 \ldots m+n} D_{:}^n b'^{m+n}.$$

It may be worth while to give an instance of the truth of the equation $D_{:}^m D_x^m b'^n = D_y^m b'^n$. Let us construct $D_{:}^3 D_x^3 b'^4$,

$$D_x b'^4 = 4b'^3 c'',$$
$$D_x^2 b'^4 = 4b'^3 e''' + 6b'^2 c''^2,$$
$$D_x^3 b'^4 = 4b'^3 f^{iv} + 12 b'^2 c'' e''' + 4b' c''^3.$$

We cannot now use the rule of the last or last but one, but must proceed with every letter, dropping terms already obtained. I begin from the last letters in each term:

$$D_{:} D_x^3 b'^4 = 4b'^3 f_{,}''' + 12 b'^2 b_{,} f^{iv} + 12 b'^2 c'' e_{,}'' + 12 b'^2 c_{,}' e''' + 24 b' b_{,} c'' e''' + 12 b' c''^2 c_{,}' + 4 b_{,} c''^3$$

$$D_{:}^2 D_x^3 b'^4 = 4b'^3 f_{,,}'' + 12 b'^2 b_{,} f_{,}''' + 12 b' b_{,}^2 f^{iv} + 12 b'^2 c'' e_{,,}' + 12 b'^2 c_{,}' e_{,}'' + 24 b' b_{,} c'' e_{,}'' + 12 b'^2 c_{,,} e''' + 24 b' b_{,} c_{,}' e''' + 12 b_{,}^2 c'' e''' + 12 b' c''^2 c_{,,} + 12 b' c'' c_{,}'^2 + 12 b_{,} c''^2 c_{,}'$$

$$D_{:}^3 D_x^3 b'^4 = 4b'^3 f'_{,,,} + 12 b'^2 b_{,} f_{,,}'' + 12 b' b_{,}^2 f_{,}''' + 4 b_{,}^3 f^{iv} + 12 b'^2 c'' e_{,,,} + 12 b'^2 c_{,}' e_{,,}' + 24 b' b_{,} c'' e_{,,}' + 12 b'^2 c_{,,} e_{,}'' + 24 b' b_{,} c_{,}' e_{,}'' + 12 b_{,}^2 c'' e_{,}'' + 24 b' b_{,} c_{,,} e''' + 12 b_{,}^2 c_{,}' e''' + 12 b_{,}^2 c'' e_{,}'' + 12 b_{,}^2 c_{,}' e''' + 24 b' c'' c_{,}' c_{,,} + 12 b_{,} c'^2 c_{,,} + 4 b' c_{,}'^3 + 12 b_{,} c'' c_{,}'^2.$$

Now this term occurs in $D_{:}^3 D_x^7 \phi a$, the coefficient of $x^4 y^3$, in which it is the coefficient of $\phi^{iv} a \div 2.3.4$.

But $$D^3_{y:x} D_x^7 \phi a = D^4_{x:y} D_y^7 \phi a;$$

in which last the coefficient of $\phi^{iv} a \div 2.3.4$ is $D_{:}^4 D_y^3 b_{,}^4$, which is therefore $= D_{:}^3 D_x^3 b'^4$. And thus we have, generally,

$$D^m_{y \cdot x} D_x^n b'^p = D^{n+p-m}_{x:y} D_y^n b_{,}^p.$$

Now $D_y^3 b_{,}^4$ is only $D_x^3 b'^4$ with superfixes and suffixes interchanged; and $D_{:}^4 D_y^3 b_{,}^4$ will only be $D_{:}^4 D_x^3 b'^4$ with similar

x^4, x^3y, x^2y^2, xy^3, y^4, are

$f^{\text{iv}}\phi_1 + (2b'e''' + c''^2)\,\phi_2 + 3b'^2c''\phi_3 + b'^4\phi_4$

$f_{,}'''\phi_1 + (2b'e_{,}'' + 2b_{,}e''' + 2c''c_{,}')\,\phi_2 + (3b'^2c_{,}' + 6b'b_{,}c'')\,\phi_3 + 4b'^3b_{,}\phi_4$

$f_{,,}''\phi_1 + (2b'e_{,,}' + 2b_{,}e_{,}'' + 2c''c_{,,} + c_{,}'^2)\,\phi_2 + (3b'^2c_{,,} + 6b'b_{,}c_{,}' + 3b_{,}^2c'')\,\phi_3 + 6b'^2b_{,}^2\phi_4$

$f_{,,,}'\phi_1 + (2b'e_{,,,} + 2b_{,}e_{,,}' + 2c_{,}'c_{,,})\,\phi_2 + (6b'b_{,}c_{,,} + 3b_{,}^2c_{,}')\,\phi_3 + 4b'b_{,}^3\phi_4$

$f_{\text{iv}}\phi_1 + (2b_{,}e_{,,,} + c_{,,}^2)\,\phi_2 + 3b_{,}^2c_{,,}\phi_3 + b_{,}^4\phi_4.$

The coefficients of x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5, are

$g^{\text{v}}\phi_1 + (2b'f^{\text{iv}} + 2c''e''')\,\phi_2 + (3b'^2e''' + 3b'c''^2)\,\phi_3 + 4b'^3c''\phi_4 + b'^5\phi_5$

$g_{,}^{\text{iv}}\phi_1 + (2b'f_{,}''' + 2b_{,}f^{\text{iv}} + 2c''e_{,}'' + 2c_{,}'e''')\,\phi_2 + (3b'^2e_{,}'' + 6b'b_{,}e''' + 6b'c''c_{,}' + 3b_{,}c''^2)\,\phi$
$\quad + (4b'^3c_{,}' + 12b'^2b_{,}c'')\,\phi_4 + 5b'^4b_{,}\phi_5$

$g_{,,}'''\phi_1 + (2b'f_{,,}'' + 2b_{,}f_{,}''' + 2c''e_{,,}' + 2c_{,}'e_{,}'' + 2c_{,,}e''')\,\phi_2$
$\quad + (3b'^2e_{,,}' + 6b'b_{,}e_{,}'' + 3b_{,}^2e''' + 6b'c''c_{,,} + 3b'c_{,}'^2 + 6b_{,}c''c_{,}')\,\phi_3$
$\quad + (4b'^3c_{,,} + 12b'^2b_{,}c_{,}' + 12b'b_{,}^2c'')\,\phi_4 + 10b'^3b_{,}^2\phi_5$

$g_{,,,}''\phi_1 + (2b'f'_{,,,} + 2b_{,}f_{,}'' + 2c''e_{,,,} + 2c_{,}'e_{,,}' + 2c_{,,}e_{,}'')\,\phi_2$
$\quad + (3b'^2e_{,,,} + 6b'b_{,}e_{,,}' + 3b_{,}^2e_{,}'' + 6b'c_{,}'c_{,,} + 6b_{,}c''c_{,,} + 3b_{,}c_{,}'^2)\,\phi_3$
$\quad + (12b'^2b_{,}c_{,,} + 12b'b_{,}^2c_{,}' + 4b_{,}^3c'')\,\phi_4 + 10b'^2b_{,}^3\phi_5$

$g'_{\text{iv}}\phi_1 + (2b'f_{\text{iv}} + 2b_{,}f'_{,,,} + 2c_{,}'e_{,,,} + 2c_{,,}e'_{,,})\,\phi_2 + (6b'b_{,}e_{,,,} + 3b_{,}^2e'_{,,} + 3b'c_{,,}^2 + 6b_{,}c_{,}'c_{,,})\,\phi_3$
$\quad + (12b'b_{,}^2c_{,,} + 4b_{,}^3c_{,}')\,\phi_4 + 5b'b_{,}^4\phi_5$

$g_{\text{v}}\phi_1 + (2b_{,}f_{\text{iv}} + 2c_{,,}e_{,,,})\,\phi_2 + (3b_{,}^2e_{,,,} + 3b_{,}c_{,,}^2)\,\phi_3 + 4b_{,}^3c_{,,}\phi_4 + b_{,}^5\phi_5.$

§ 4. *On Functions of two series of one Variable.*

Having $V = \phi(a, A)$, it is proposed to develope

$$\phi(a + bx + cx^2 + \ldots,\quad A + Bx + Cx^2 + \ldots).$$

If we write this with two different variables, as

$$\phi(a + bx + \ldots,\quad A + By + \ldots),$$

and if we first write $a + bx + \ldots$ for a in $\phi(a, A)$ or V, the development is $V + D_xV.x + D_x^2V.x^2 + \ldots$ If in each of the functions V, D_xV, &c. we write $A + By + \ldots$ for A, we find for the development required,

$$V + D_xV.x + D_yV.y + D_x^2V.x^2 + D_xD_yV.xy + D_y^2V.y^2 + \ldots$$

Change y into x, and the coefficient of x^n is

$$D_x^nV + D_x^{n-1}D_yV + \ldots + D_xD_y^{n-1}V + D_y^nV.$$

If the sum of the partial derivatives with respect to a, b, &c. and A, B, &c. be called the total derivative, and denoted by $D_{,,}$, we have

$$\phi(a, A) + D_{,,}\phi(a, A)\,x + D_{,,}^2\phi(a, A).x^2 + \ldots$$

for the development. To expand this form, observe that in

$$D_x^m\phi(a, A) = \phi' D^{m-1} \ldots \text{[illegible]} \ldots \frac{\phi^{(m)}}{1 \ldots m}\,b'^m,$$

[redacted]

ON SYMBOLICAL GEOMETRY.

By Sir William Hamilton.

[*Continued.*]

On the Distributive Character of the Operation of Multiplication, as performed generally on Geometrical Fractions.

14. We are now prepared to extend the formulæ (76), (77), respecting the multiplication of sums of geometrical fractions; and to shew that similar results hold good, even when the condition of colinearity, assumed in those two formulæ, is no longer supposed to be satisfied. That is, the two equations

$$\left(\frac{\mathrm{h}}{\mathrm{g}} + \frac{\mathrm{f}}{\mathrm{e}}\right) \times \frac{\mathrm{k}}{\mathrm{i}} = \left(\frac{\mathrm{h}}{\mathrm{g}} \times \frac{\mathrm{k}}{\mathrm{i}}\right) + \left(\frac{\mathrm{f}}{\mathrm{e}} \times \frac{\mathrm{k}}{\mathrm{i}}\right) \ldots\ldots (104),$$

$$\frac{\mathrm{k}}{\mathrm{i}} \times \left(\frac{\mathrm{h}}{\mathrm{g}} + \frac{\mathrm{f}}{\mathrm{e}}\right) = \left(\frac{\mathrm{k}}{\mathrm{i}} \times \frac{\mathrm{h}}{\mathrm{g}}\right) + \left(\frac{\mathrm{k}}{\mathrm{i}} \times \frac{\mathrm{f}}{\mathrm{e}}\right) \ldots\ldots (105),$$

can both be shown to be true, whatever may be the lengths and directions of the six lines e, f, g, h, i, k; although, by the general non-commutativeness of geometrical fractions as factors, which was pointed out in the last article, the expressions contained in these two equations are not to be confounded with each other.

Making for this purpose

$$\left.\begin{array}{l} \frac{\mathrm{f}}{\mathrm{e}} = \beta_1 + b_1, \quad \frac{\mathrm{h}}{\mathrm{g}} = \beta_2 + b_2, \quad \frac{\mathrm{k}}{\mathrm{i}} = \alpha + a, \\ \mathrm{I}\beta_1' \parallel \mathrm{I}\alpha, \quad \mathrm{I}\beta_1'' \perp \mathrm{I}\alpha, \quad \mathrm{I}\beta_1'' + \mathrm{I}\beta_1' = \mathrm{I}\beta_1, \\ \mathrm{I}\beta_2' \parallel \mathrm{I}\alpha, \quad \mathrm{I}\beta_2'' \perp \mathrm{I}\alpha, \quad \mathrm{I}\beta_2'' + \mathrm{I}\beta_2' = \mathrm{I}\beta_2, \\ \beta_2' + \beta_1' = \beta', \ \beta_2'' + \beta_1'' = \beta'', \ \beta_2 + \beta_1 = \beta, \ b_2 + b_1 = b, \end{array}\right\} \ldots (106),$$

the conditions (83) will be satisfied; and if we still assign to γ and c the meanings (87), the equation (88) will hold good, and $\gamma + c$ will be an expression for the first member of (104). Making also, in imitation of (87),

$$\left.\begin{array}{l} c_1 = \beta_1'\alpha + b_1 a, \quad \gamma_1 = \beta_1''\alpha + \beta_1 a + b_1\alpha, \\ c_2 = \beta_2'\alpha + b_2 a, \quad \gamma_2 = \beta_2''\alpha + \beta_2 a + b_2\alpha, \end{array}\right\} \ldots (107),$$

the second member of the same equation (104) becomes, by the principles of the 11th article, $(\gamma_2 + c_2) + (\gamma_1 + c_1)$; and the equation resolves itself into the two following,

$$c = c_2 + c_1, \quad \gamma = \gamma_2 + \gamma_1 \ldots\ldots\ldots (108);$$

which are easily seen to reduce themselves to these two,

$$(\beta_2' + \beta_1')\,\alpha = \beta_2'\alpha + \beta_1'\alpha; \quad (\beta_2'' + \beta_1'')\,\alpha = \beta_2''\alpha + \beta_1''\alpha \ldots (109);$$

[redacted]

ON THE ROTATION OF A SOLID BODY ROUND A FIXED POINT.

By Arthur Cayley.

(*Continued from* p. 173.)

On the Variation of the Constants, when the body is acted upon by forces.

The dynamical equations of a problem being expressed in the form

$$\frac{d}{dt}\cdot\frac{dT}{d\lambda'}-\frac{dT}{d\lambda}=\frac{dV}{d\lambda},$$

$$\frac{d}{dt}\cdot\frac{dT}{d\mu'}-\frac{dT}{d\mu}=\frac{dV}{d\mu}.$$

$$\frac{d}{dt}\cdot\frac{dT}{d\nu'}-\frac{dT}{d\nu}=\frac{dV}{d\nu}.$$

Suppose the equations obtained from these by neglecting the function V, are integrated; each of the six integrals may be expressed in the form

$$a=f(\lambda,\ \mu,\ \nu,\ \lambda',\ \mu',\ \nu',\ t),$$

where a denotes any one of the arbitrary constants. Assume

$$\frac{dT}{d\lambda'}=u,\quad \frac{dT}{d\mu'}=v,\quad \frac{dT}{d\nu}=w.$$

Then λ', μ', ν' may be expressed in terms of λ, μ, ν, u, v, w, and the integrals may be reduced to the form

$$a=F(\lambda,\ \mu,\ \nu,\ u,\ v,\ w,\ t).$$

These equations may be considered as the integrals of the proposed system, taking into account the terms involving V, provided a, b, ... &c. be supposed to become variable. We have, in this case, by Lagrange's theory of the variation of the arbitrary constants, the formulæ

$$\frac{da}{dt}=(a,b)\frac{dV}{db}+(a,c)\frac{dV}{dc}+(a,d)\frac{dV}{dd}+(a,e)\frac{dV}{de}+(a,f)\frac{dV}{df};$$

where

$$(a,b)=\left(\frac{da}{du}\frac{db}{d\lambda}-\frac{da}{d\lambda}\frac{db}{du}\right)+\left(\frac{da}{dv}\frac{db}{d\mu}-\frac{da}{d\mu}\frac{db}{dv}\right)+\left(\frac{da}{dw}\frac{db}{d\nu}-\frac{da}{d\nu}\frac{db}{dw}\right),$$

and in which V is supposed to be expressed as a function of a, b, c, d, e, f, t.

$$2\mathrm{a} = \lambda\varpi + u - \nu v + \mu w \ldots\ldots\ldots\ldots(34),$$
$$2\mathrm{b} = \mu\varpi + \nu u + v - \lambda w,$$
$$2\mathrm{c} = \nu\varpi - \mu u + \lambda v + w,$$

whence also $\qquad 2\,(\mathrm{a}\lambda + \mathrm{b}\mu + \mathrm{c}\nu) = \kappa\varpi \ldots\ldots\ldots\ldots(35),$

which in fact follows from (33) and (17). And likewise the inverse system,

$$u = \frac{2}{\kappa}\,(\mathrm{a} + \nu\mathrm{b} - \mu\mathrm{c}) \ldots\ldots\ldots\ldots(36),$$

$$v = \frac{2}{\kappa}\,(-\,\nu\mathrm{a} + \mathrm{b} + \lambda\mathrm{c}),$$

$$w = \frac{2}{\kappa}\,(\mu\mathrm{a} - \lambda\mathrm{b} + \mathrm{c}).$$

It is easy to deduce

$$k^2 = \frac{1}{4}\,\kappa\,[u^2 + v^2 + w^2 + \varpi^2] \ldots\ldots\ldots\ldots(37),$$

$$v = \frac{1}{4}\,[(u^2 + v^2 + w^2) + (1 + \kappa)\,\varpi^2] \ldots\ldots(38).$$

Again, from the equations (10 *bis*),

$$\begin{aligned}\kappa(\mathrm{b}Cr - \mathrm{c}Bq) &= -\,2\lambda\,(\mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2) + 2\mathrm{a}\,(\lambda\mathrm{a} + \mu\mathrm{b} + \nu\mathrm{c}) + 2\,(\mathrm{b}\nu - \mathrm{c}\mu)\,\Omega \\ &= -\,2\lambda k^2 + 2\,(\mathrm{a} + \mathrm{b}\nu - \mathrm{c}\mu)\,\Omega \\ &= -\,2\lambda k^2 + \kappa u\,\Omega\,;\end{aligned}$$

[by equations (36),] *i.e.*

$$\Omega u - \frac{2}{\kappa}\,k^2\lambda = \mathrm{b}Cr - \mathrm{c}Bq \ldots\ldots\ldots\ldots(39),$$

$$\Omega v - \frac{2}{\kappa}\,k^2\mu = \mathrm{c}Ap - \mathrm{b}Cr,$$

$$\Omega w - \frac{2}{\kappa}\,k^2\nu = \mathrm{a}Bq - \mathrm{c}Ap\,;$$

to which many others might probably be joined.

The constants of the problem are a, b, c, h, ϵ, δ. Of these a, b, c are given as functions of λ, μ, ν, u, v, w, by the equations (34); in which ϖ is to be considered as standing for $\lambda u + \mu v + \nu w$. [These determine k^2, which is however given immediately by (37).] As for h, we have

$$h = \frac{1}{A}(Ap)^2 + \frac{1}{B}(Bq)^2 + \frac{1}{C}(Cr)^2 \ldots\ldots(40),$$

Similarly

$$(\mathrm{a}, Bq) = \tfrac{1}{4}\{(1+\lambda^2)(\mu u + w) - (\lambda u + \varpi)(\lambda\mu - \nu) + (\lambda\mu - \nu)(\mu v + \varpi) - (\lambda v + w)(1+\mu^2) + (\nu\lambda + \mu)(\mu w - u) - (-v + \lambda w)(\mu\nu + \lambda)\}$$

i.e. $(\mathrm{a}, Bq) = 0$, and similarly ...... (43),

$(\mathrm{a}, Cr) = 0$;

whence $(\mathrm{a}, h) = 0$, and $\therefore (\mathrm{b}, h) = 0$, $(\mathrm{c}, h) = 0$ ...... (44);

also $(k, h) = 0$, ... (45).

Next we have to determine (a, ϵ), (b, ϵ), (c, ϵ). Here ϵ being a function of $u, v, w, \lambda, \mu, \nu$, a, b, c, h, we must write

$$(\mathrm{a}, \epsilon) = \{(\mathrm{a}, \epsilon)\} + (\mathrm{a}, \mathrm{b})\frac{d\epsilon}{d\mathrm{b}} + (\mathrm{a}, \mathrm{c})\frac{d\epsilon}{d\mathrm{c}} + (\mathrm{a}, h)\frac{d\epsilon}{dh},$$

i.e. $$(\mathrm{a}, \epsilon) = \{(\mathrm{a}, \epsilon)\} + \mathrm{b}\frac{d\epsilon}{d\mathrm{c}} - \mathrm{c}\frac{d\epsilon}{d\mathrm{b}}.$$

But $$\epsilon = t - 2\int\frac{dv}{\nabla};$$

whence $$\{(\mathrm{a}, \epsilon)\} = -\frac{2}{\nabla}(\mathrm{a}, v),$$

and v is given immediately as a function of $\lambda, \mu, \nu, u, v, w$, by the equation (38). Hence

$$\begin{aligned}(\mathrm{a}, v) &= \tfrac{1}{4}[(1+\lambda^2)\{(1+\kappa)u\varpi + \lambda\varpi^2\} - (\lambda u + \varpi)\{u + \lambda(1+\kappa)\varpi\} \\ &\quad + (\lambda\mu - \nu)\{(1+\kappa)v\varpi + \mu\varpi^2\} - (\lambda v + w)\{v + \mu(1+\kappa)\varpi\} \\ &\quad + (\nu\lambda + \mu)\{(1+\kappa)w\varpi + \nu\varpi^2) - (-v + \lambda w)\{w + \nu(1+\kappa)\varpi\}] \\ &= \tfrac{1}{4}\{(1+\kappa)\varpi u - \lambda(1+\kappa)\varpi^2 + \lambda\kappa - \lambda(u^2+v^2+w^2) - u\varpi\} \\ &= \tfrac{1}{4}\{\kappa u\varpi - \lambda\varpi^2 - \lambda(u^2+v^2+w^2)\} \\ &= \tfrac{1}{4}\kappa u\varpi - \frac{k^2\lambda}{\kappa} = \tfrac{1}{2}\left(\Omega u - \frac{2k^2\lambda}{\kappa}\right) \text{ [by (37) and (33)]}, \\ &= \tfrac{1}{2}(\mathrm{b}Cr - \mathrm{c}Bq). \qquad \text{...... (46);}\end{aligned}$$

whence $$\{(\mathrm{a}, \epsilon)\} = -\frac{1}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq),$$

$$(\mathrm{a}, \epsilon) = -\frac{1}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) + \mathrm{b}\frac{d\epsilon}{d\mathrm{c}} - \mathrm{c}\frac{d\epsilon}{d\mathrm{b}}.$$

The terms $\mathrm{b}\dfrac{d\epsilon}{d\mathrm{c}} - \mathrm{c}\dfrac{d\epsilon}{d\mathrm{b}}$ are evidently of the form $F(v) - F(v_0)$. If therefore we suppose $v = v_0$, we have

$$(\mathrm{a}, \epsilon) = -\frac{1}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0) \text{ .... (47)},$$

if p_0, q_0, r_0, ∇_0 refer to the value v_0 of v, i.e. if

$$Ap_0^2 + Bq_0^2 + Cr_0^2 = h \qquad\qquad (48),$$
$$A^2p_0^2 + B^2q_0^2 + C^2r_0^2 = k^2,$$
$$Ap_0\mathrm{a} + Bq_0\mathrm{b} + Cr_0\mathrm{c} = 2v_0 - k^2.$$

[This implies evidently

$$\mathrm{b}\frac{d\epsilon}{d\mathrm{c}} - \mathrm{c}\frac{d\epsilon}{d\mathrm{b}} = \frac{1}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) - \frac{1}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0),$$

an equation which it is interesting to verify. In fact, from the value of ϵ

$$\mathrm{b}\frac{d\epsilon}{d\mathrm{c}} - \mathrm{c}\frac{d\epsilon}{d\mathrm{b}} = -2\int dv\left(\mathrm{b}\frac{d}{d\mathrm{c}} - \mathrm{c}\frac{d}{d\mathrm{b}}\right)\frac{1}{\nabla} = 2\int dv\,\frac{1}{\nabla^2}\left(\mathrm{b}\frac{d\nabla}{d\mathrm{c}} - \mathrm{c}\frac{d\nabla}{d\mathrm{b}}\right);$$

or we have to shew that

$$\frac{d}{dv}\frac{1}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) = \frac{2}{\nabla^2}\left(\mathrm{b}\frac{d\nabla}{d\mathrm{c}} - \mathrm{c}\frac{d\nabla}{d\mathrm{b}}\right) = \frac{2}{\nabla^2}\delta\nabla;$$

if for shortness,

$$\delta = \mathrm{b}\frac{d}{d\mathrm{c}} - \mathrm{c}\frac{d}{d\mathrm{b}}.$$

Now ∇ containing a, b, c explicitly, and also as involved in p, q, r, we have

$$\delta\nabla = \mathrm{b}pq(A - B) - \mathrm{c}rp(C - A) + \frac{d\nabla}{dp}\delta p + \frac{d\nabla}{dq}\delta q + \frac{d\nabla}{dr}\delta r$$
$$= \mathrm{b}pq(A - B) - \mathrm{c}rp(C - A) + \delta'\nabla$$

suppose. The equation to be verified becomes

$$\nabla\left(\mathrm{b}C\frac{dr}{dv} - \mathrm{c}B\frac{dq}{dv}\right) - (\mathrm{b}Cr - \mathrm{c}Bq)\frac{d\nabla}{dv}$$
$$= 2\{\mathrm{b}pq(A - B) - \mathrm{c}rp(C - A) + \delta'\nabla\}.$$

Now, observing that $\delta k = 0$, we have

$$Ap\delta p + Bq\delta q + Cr\delta r = 0,$$
$$A^2p\delta p + B^2q\delta q + C^2r\delta r = 0,$$
$$A\mathrm{a}\delta p + B\mathrm{b}\delta q + C\mathrm{c}\delta r = -(\mathrm{b}Cr - \mathrm{c}Bq).$$

Also,
$$Ap\frac{dp}{dv} + Bq\frac{dq}{dv} + Cr\frac{dr}{dv} = 0,$$
$$A^2p\frac{dp}{dv} + B^2q\frac{dq}{dv} + C^2r\frac{dr}{dv} = 0,$$
$$A\mathrm{a}\frac{dp}{dv} + B\mathrm{b}\frac{dq}{dv} + C\mathrm{c}\frac{dr}{dv} = 2.$$

[redacted]

Whence evidently

$$\frac{dp}{dv} = \frac{-2}{\mathrm{b}Cr - \mathrm{c}Bq}\,\delta p,\quad \frac{dq}{dv} = \frac{-2}{\mathrm{b}Cr - \mathrm{c}Bq}\,\delta q,\quad \frac{dr}{dv} = \frac{-2}{\mathrm{b}Cr - \mathrm{c}Bq}\,\delta r,$$

or

$$\frac{d\nabla}{dv} = \frac{-2}{\mathrm{b}Cr - \mathrm{c}Bq}\,\delta'\nabla\,;$$

or the equation to be verified is simply

$$\nabla\left(\mathrm{b}C\frac{dr}{dv} - \mathrm{c}B\frac{dq}{dv}\right) = 2\,\{\mathrm{b}pq\,(A-B) - \mathrm{c}rp\,(C-A)\}\,;$$

which follows immediately from the three equations just given for the determination of $\frac{dp}{dv}$, $\frac{dq}{dv}$, $\left.\frac{dr}{dv}\right]$.

From which values also

$$(k,\ \epsilon) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (49).$$

Next, to calculate $(h,\ \epsilon)$,

$$(h,\ \epsilon) = \{(h,\ \epsilon)\} + (h,\ \mathrm{a})\frac{d\epsilon}{d\mathrm{a}} + (h,\ \mathrm{b})\frac{d\epsilon}{d\mathrm{b}} + (h,\ \mathrm{c})\frac{d\epsilon}{d\mathrm{c}}\,.$$

But the three last terms being evidently such as to vanish for $v = v_0$, we may neglect them, and consider $(h,\ \epsilon)$ as the value which $\{(h,\ \epsilon)\}$ assumes for this value of v.

Now $\{(h,\ \epsilon)\} = 2p\,\{(Ap,\ \epsilon)\} + 2q\,\{(Bq,\ \epsilon)\} + 2r\,\{(Cr,\ \epsilon)\}$,

where $\quad \{(Ap,\ \epsilon)\} = -\frac{2}{\nabla}(Ap,\ v),$

and

$$\begin{aligned}(Ap,\ v) = \tfrac{1}{4}\big[&(1+\lambda^2)\{(1+\kappa)u\varpi + \lambda\varpi^2\} - (\lambda u + \varpi)\{u + \lambda(1+\kappa)\varpi\}\\ &+ (\lambda\mu + \nu)\{(1+\kappa)v\varpi + \mu\varpi^2\} - (\lambda v - w)\{v + \mu(1+\kappa)\varpi\}\\ &+ (\nu\lambda - \mu)\{(1+\kappa)w\varpi + \nu\varpi^2\} - (v + \lambda w)\{w + \nu(1+\kappa)\varpi\}\big]\\ = \tfrac{1}{4}\,&\{(1+\kappa)\varpi u + \lambda\kappa\varpi^2 - \lambda(1+\kappa)\varpi^2 - \lambda(u^2+v^2+w^2) - \varpi u\} = (\mathrm{a},\ v)\\ = \tfrac{1}{2}\,&(\mathrm{b}Cr - \mathrm{c}Bq), \qquad \ldots\ldots (50),\end{aligned}$$

whence $\quad \{(Ap,\ \epsilon)\} = -\frac{1}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) \ldots\ldots\ldots (51),$

and therefore $\quad \{(Bq,\ \epsilon)\} = -\frac{1}{\nabla}(\mathrm{c}Ap - \mathrm{a}Cr),$

$$\{(Cr,\ \epsilon)\} = -\frac{1}{\nabla}(\mathrm{a}Bq - \mathrm{b}Ap),$$

whence $\quad \{(h,\ \epsilon) = -2,$

and therefore $\quad (h,\ \epsilon) = -2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (52).$

Next, to find (a, δ), (b, δ), (c, δ), (h, δ),

$$\delta = 2\tan^{-1}\frac{\kappa\varpi}{2k} - k\int\frac{(h+\Phi)\,dv}{v\nabla}$$

$$= \delta' + \delta'' \quad \text{suppose},$$

$$(\mathrm{a}, \delta) = (\mathrm{a}, \delta') + (\mathrm{a}, \delta''),$$

$$(\mathrm{a}, \delta') = \frac{k}{\kappa v}(\mathrm{a}, \kappa\varpi) + (\mathrm{a}, k)\frac{d\delta'}{dk}$$

[observing $\kappa^2\varpi^2 + 4k^2 = 4(\Omega^2 + k^2) = 4\kappa v$]

$$= \frac{k}{\kappa v}(\mathrm{a}, \kappa\varpi),$$

where

$$\begin{aligned}(\mathrm{a}, \kappa\varpi) = \tfrac{1}{2}\{&(1+\lambda^2)(\kappa u + 2\lambda\varpi) - (\lambda u + \varpi)\kappa\lambda \\ &+ (\lambda\mu - \nu)(\kappa v + 2\mu\varpi) - (\lambda v + w)\kappa\mu \\ &+ (\nu\lambda + \mu)(\kappa w + 2\nu\varpi) - (-v + \lambda w)\kappa\nu\}\end{aligned}$$

$$= \tfrac{1}{2}\kappa(u + \lambda\varpi) = Ap - \nu Bq + \mu Cr + \lambda\Omega = \tfrac{1}{2}(\mathrm{a} + Ap)\kappa \ldots (53),$$

by equations (29), (33), and (10);

or $$(\mathrm{a}, \delta') = \frac{k}{2v}(\mathrm{a} + Ap).$$

Also $$(\mathrm{a}, \delta'') = -k\frac{h+\Phi}{v\nabla}(\mathrm{a}, v) + (\mathrm{a}, \mathrm{b})\frac{d\delta''}{d\mathrm{b}} + \&\mathrm{c}.$$

$$= -\tfrac{1}{2}k\frac{h+\Phi}{v\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) + Fv - Fv_0,$$

whence

$$(\mathrm{a}, \delta) = \frac{k}{2v}\left\{\mathrm{a} + Ap - \frac{h+\Phi}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq)\right\} + Fv - Fv_0,$$

or putting $v = v_0$,

$$(\mathrm{a}, \delta) = \frac{k}{2v_0}\left\{\mathrm{a} + Ap_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\right\} \ldots\ldots (54),$$

and therefore

$$(\mathrm{b}, \delta) = \frac{k}{2v_0}\left\{\mathrm{b} + Bq_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0)\right\},$$

$$(\mathrm{c}, \delta) = \frac{k}{2v_0}\left\{\mathrm{c} + Cr_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\right\}.$$

Again, $(h, \delta) = 2p(Ap, \delta) + 2q(Bq, \delta) + 2r(Cr, \delta)$,

$$(Ap, \delta) = (Ap, \delta') + (Ap, \delta''),$$

$$(Ap, \delta') = \frac{k}{\kappa v}(Ap, \kappa\varpi) + (Ap, k)\frac{d\delta'}{dk}$$

$$= \frac{k}{\kappa v}(Ap, \kappa\varpi),$$

$$\begin{aligned}(Ap, \kappa\varpi) = \tfrac{1}{2}\{&(1+\lambda^2)(\kappa u + 2\lambda\varpi) - (\lambda u + \varpi)\kappa\lambda \\ &+ (\lambda\mu + \nu)(\kappa v + 2\mu\varpi) - (\lambda v - w)\kappa\mu \\ &+ (\nu\lambda + \mu)(\kappa w + 2\nu\varpi) - (v + \lambda w)\kappa\nu\} \\ = \tfrac{1}{2}&\kappa(u + \lambda\varpi) = \tfrac{1}{2}\kappa(\mathrm{a} + Ap) \;\ldots\ldots\ldots\; (55);\end{aligned}$$

$$\therefore (Ap, \delta') = \frac{k}{2u}(\mathrm{a} + Ap) \;\ldots\ldots\ldots\ldots\ldots\; (56),$$

$$\begin{aligned}(Ap, \delta'') &= -k\frac{h+\Phi}{v\nabla}(Ap, v) + \&\mathrm{c}. \\ &= -\tfrac{1}{2}k\frac{h+\Phi}{v\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) + Fv - Fv_0,\end{aligned}$$

$$(Ap, \delta) = \frac{k}{2v}\left\{\mathrm{a} - Ap - \frac{h+\Phi}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq)\right\} + Fv - Fv_0,$$

and similarly for (Bq, δ), (Cr, δ). Substituting, and neglecting the terms which vanish for $v = v_0$,

$$(h, \delta) = \frac{k}{v}\left(\Phi + h - \frac{\Phi + h}{\nabla}\nabla\right),$$

$$\text{i.e.}\quad (h, \delta) = 0 \;\ldots\ldots\ldots\ldots\ldots\; (57).$$

Lastly, to find (ϵ, δ),

$$(\epsilon, \delta) = \{(\epsilon, \delta)\} + (\mathrm{a}, \delta)\frac{d\epsilon}{d\mathrm{a}} + (\mathrm{b}, \delta)\frac{d\epsilon}{d\mathrm{b}} + (\mathrm{c}, \delta)\frac{d\epsilon}{d\mathrm{c}},$$

where, in $\{(\epsilon, \delta)\}$, the differentiations upon ϵ are supposed not to affect the constants a, b, c. Neglecting the terms which vanish for $v = v_0$,

$$\begin{aligned}(\epsilon, \delta) &= \{(\epsilon, \delta)\}, \\ \{(\epsilon, \delta)\} &= \{(\epsilon, \delta')\} + \{(\epsilon, \delta'')\}, \\ \{(\epsilon, \delta')\} &= [\{(\epsilon, \delta')\}] + (\epsilon, k)\frac{d\delta'}{dk} = [\{(\epsilon, \delta')\}];\end{aligned}$$

where, in $[\{(\epsilon, \delta')\}]$, the differentiations upon ϵ and δ do not affect the constants.

$$\{(\epsilon, \delta'')\} = [\{(\epsilon, \delta'')\}] + (\epsilon, \mathrm{a})\frac{d\delta''}{d\mathrm{a}} + \&\mathrm{c}.$$

$$\text{i.e.}\quad \{(\epsilon, \delta'')\} = [\{(\epsilon, \delta'')\}]:$$

neglecting the terms which vanish for $v = v_0$,

$$\begin{aligned}\therefore (\epsilon, \delta) &= [\{(\epsilon, \delta')\}] + [\{(\epsilon, \delta'')\}] \\ &= [\{(\epsilon, \delta')\}];\end{aligned}$$

since $$[\{(\epsilon, \delta'')\}] = (\upsilon, \upsilon)\,\frac{d\epsilon}{d\upsilon}\,\frac{d\delta''}{d\upsilon} = 0.$$

Hence $$(\epsilon, \delta) = -\tfrac{1}{2}\,\frac{k}{\kappa\nabla\upsilon}\,(\upsilon, \kappa\varpi)\,\ldots\ldots\ldots\ldots(58),$$

$$(\upsilon, \kappa\varpi) = \tfrac{1}{2}\left\{\begin{aligned} &\{u+(1+\kappa)\lambda\varpi\}\,(2\lambda\varpi+\kappa u)-\{\lambda\varpi^2+(1+\kappa)\varpi u\}\,\kappa\lambda \\ +&\{v+(1+\kappa)\mu\varpi\}\,(2\mu\varpi+\kappa v)-\{\mu\varpi^2+(1+\kappa)\varpi v\}\,\kappa\mu \\ +&\{w+(1+\kappa)\nu\varpi\}\,(2\nu\varpi+\kappa w)-\{\nu\varpi^2+(1+\kappa)\varpi w\}\,\kappa\nu \end{aligned}\right\}$$

$$\begin{aligned} &= \tfrac{1}{2}\{2\varpi^2+\kappa(u^2+v^2+w^2)+2(1+\kappa)(\kappa-1)\varpi^2 \\ &\qquad +\kappa(1+\kappa)\varpi^2-\kappa(\kappa-1)\varpi^2-\kappa(\kappa+1)\varpi^2\} \\ &= \tfrac{1}{2}\kappa\{(\kappa+1)\varpi^2+(u^2+v^2+w^2)\} = \tfrac{1}{2}\,4\kappa\upsilon = 2\kappa\upsilon\,\ldots(59), \end{aligned}$$

therefore
$$(\epsilon, \delta) = -\frac{k}{\nabla_0}\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(60).$$

Hence, recapitulating,

$$\left.\begin{aligned} &(\mathrm{b}, \mathrm{c}) = -\mathrm{a}, \quad (\mathrm{c}, \mathrm{a}) = -\mathrm{b}, \quad (\mathrm{a}, \mathrm{b}) = -\mathrm{c}, \\ &(\mathrm{a}, h) = 0, \quad (\mathrm{b}, h) = 0, \quad (\mathrm{c}, h) = 0, \\ &(\mathrm{a}, \epsilon) = -\frac{1}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0), \\ &(\mathrm{b}, \epsilon) = -\frac{1}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0), \\ &(\mathrm{c}, \epsilon) = -\frac{1}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0), \\ &(h, \epsilon) = -2, \\ &(\mathrm{a}, \delta) = \frac{k}{2\upsilon_0}\left\{\mathrm{a} + Ap_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\right\}, \\ &(\mathrm{b}, \delta) = \frac{k}{2\upsilon_0}\left\{\mathrm{b} + Bq_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0)\right\}, \\ &(\mathrm{c}, \delta) = \frac{k}{2\upsilon_0}\left\{\mathrm{c} + Cr_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\right\}, \\ &(h, \delta) = 0, \\ &(\epsilon, \delta) = -\frac{k}{\nabla_0}, \end{aligned}\right\}\ldots(61),$$

[redacted]

and therefore

$$\left.\begin{aligned}
\frac{d\mathrm{a}}{dt} &= -\mathrm{c}\,\frac{dV}{d\mathrm{b}} + \mathrm{b}\,\frac{dV}{d\mathrm{c}} - \frac{1}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\,\frac{dV}{d\epsilon} \\
&\quad + \frac{k}{2v_0}\left\{\mathrm{a} + Ap_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\right\}\frac{dV}{d\delta}, \\
\frac{d\mathrm{b}}{dt} &= -\mathrm{a}\,\frac{dV}{d\mathrm{c}} + \mathrm{c}\,\frac{dV}{d\mathrm{a}} - \frac{1}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0)\,\frac{dV}{d\epsilon} \\
&\quad + \frac{k}{2v_0}\left\{\mathrm{b} + Bq_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0)\right\}\frac{dV}{d\delta}, \\
\frac{d\mathrm{c}}{dt} &= -\mathrm{b}\,\frac{dV}{d\mathrm{a}} + \mathrm{a}\,\frac{dV}{d\mathrm{b}} - \frac{1}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\,\frac{dV}{d\epsilon} \\
&\quad + \frac{k}{2v_0}\left\{\mathrm{c} + Cr_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\right\}\frac{dV}{d\delta}, \\
\frac{dh}{dt} &= -2\,\frac{dV}{d\epsilon}, \\
\frac{d\epsilon}{dt} &= \frac{1}{\nabla_0}\left\{(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\frac{dV}{d\mathrm{a}} + (\mathrm{c}Ap_0 - \mathrm{a}Bq_0)\frac{dV}{d\mathrm{b}}\right. \\
&\quad \left. + (\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\frac{dV}{d\mathrm{c}}\right\} + 2\,\frac{dV}{dh} - \frac{k}{\nabla_0}\,\frac{dV}{d\delta}, \\
\frac{d\delta}{dt} &= -\frac{k}{2v_0}\left[\left\{\mathrm{a} + Ap_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\right\}\frac{dV}{d\mathrm{a}}\right. \\
&\quad + \left\{\mathrm{b} + Bq_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0)\right\}\frac{dV}{d\mathrm{b}} \\
&\quad \left. + \left\{\mathrm{c} + Cr_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\right\}\frac{dV}{d\mathrm{c}}\right] + \frac{k}{\nabla_0}\,\frac{dV}{d\epsilon},
\end{aligned}\right\} \quad \ldots\ldots (62),$$

to which we may join $\dfrac{dk}{dt} = \dfrac{dV}{d\delta}$ .... (63).

ON THE DIAMETRAL PLANES OF A SURFACE OF THE SECOND ORDER.

By Arthur Cayley.

Let $U = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gxz + 2Hxy = 0$, be the equation of a surface of the second order referred to its centre, and let $ax + a'y + a''z = 0$ be the equation of one of its diametral planes; then, as usual,

in x, y, z; and this equation, it is evident, must belong to the three diametral planes jointly, *i.e.* Θ must be the product of three linear factors, each of which equated to zero would correspond to a diametral plane. Thus the system of diametral planes is given by

$$\Theta = \begin{vmatrix} \mathfrak{A}x + \mathfrak{H}y + \mathfrak{G}z, & Ax + Hy + Gz, & x \\ \mathfrak{H}x + \mathfrak{B}y + \mathfrak{F}z, & Hx + By + Fz, & y \\ \mathfrak{G}x + \mathfrak{F}y + \mathfrak{C}z, & Gx + Fy + Cz, & z \end{vmatrix} = 0,$$

or developing the determinant, as follows,

$$\begin{aligned}\Theta = {} & (G\mathfrak{H} - H\mathfrak{G})\, x^3 + (H\mathfrak{F} - F\mathfrak{H})\, y^3 + (F\mathfrak{G} - G\mathfrak{F})\, z^3 \\ & + \{G(\mathfrak{C} - \mathfrak{B}) - \mathfrak{G}(C - B) - (H\mathfrak{F} - F\mathfrak{H})\}\, yz^2 \\ & + \{H(\mathfrak{A} - \mathfrak{C}) - \mathfrak{H}(A - C) - (F\mathfrak{G} - G\mathfrak{F})\}\, zx^2 \\ & + \{F(\mathfrak{B} - \mathfrak{A}) - \mathfrak{F}(B - A) - (G\mathfrak{H} - H\mathfrak{G})\}\, xy^2 \\ & + \{-H(\mathfrak{C} - \mathfrak{B}) + \mathfrak{H}(C - B) + (F\mathfrak{G} - G\mathfrak{F})\}\, y^2z \\ & + \{-F(\mathfrak{A} - \mathfrak{C}) + \mathfrak{F}(A - C) + (G\mathfrak{H} - H\mathfrak{G})\}\, z^2x \\ & + \{-G(\mathfrak{B} - \mathfrak{A}) + \mathfrak{C}(B - A) + (H\mathfrak{F} - F\mathfrak{H})\}\, x^2y \\ & + (C\mathfrak{B} - B\mathfrak{A} + \mathfrak{A}\mathfrak{C} - C\mathfrak{A} + B\mathfrak{A} - A\mathfrak{B})\, xyz;\end{aligned}$$

or reducing

$$\begin{aligned}\Theta = {} & \{F(G^2 - H^2) - GH(C - B)\}\, x^3 \\ & + \{G(H^2 - F^2) - HF(A - C)\}\, y^3 \\ & + \{H(F^2 - G^2) - FG(B - A)\}\, z^3 \\ & + \{G(A - B)(B - C) + FH(A + B - 2C) \\ & \qquad + G(F^2 + G^2 - 2H^2)\}\, yz^2 \\ & + \{H(B - C)(C - A) + GF(B + C - 2A) \\ & \qquad + H(G^2 + H^2 - 2F^2)\}\, zx^2 \\ & + \{F(C - A)(A - B) + GH(C + A - 2B) \\ & \qquad + F(H^2 + F^2 - 2G^2)\}\, xy^2 \\ & + \{H(B - C)(C - A) + FG(C + A - 2B) \\ & \qquad + H(H^2 + F^2 - 2G^2)\}\, y^2z \\ & + \{F(C - A)(A - B) + GH(A + B - 2C) \\ & \qquad + F(F^2 + G^2 - 2H^2)\}\, z^2x \\ & + \{G(A - B(B - C) + HF(B + C - 2A) \\ & \qquad + G(G^2 + H^2 - 2F^2)\}\, x^2y \\ & - \{(A - B)(B - C)(C - A) \\ & \qquad + (B - C)F^2 + (C - A)G^2 + (A - B)H^2\}\, xyz.\end{aligned}$$

whence
$$\begin{aligned}
A &= \beta^2c^2 + \gamma^2b^2 - b^2c^2,\\
B &= \gamma^2a^2 + \alpha^2c^2 - a^2c^2,\\
C &= \alpha^2b^2 + \beta^2a^2 - b^2a^2,\\
F &= -\,a^2\beta\gamma,\\
G &= -\,b^2\gamma\alpha,\\
H &= -\,c^2\alpha\beta.
\end{aligned}$$

And thence omitting the factor $b^2c^2\alpha^2 + c^2a^2\beta^2 + a^2b^2\gamma^2 - a^2b^2c^2$,
$$\begin{aligned}
\mathfrak{A} &= \alpha^2 - a^2,\\
\mathfrak{B} &= \beta^2 - b^2,\\
\mathfrak{C} &= \gamma^2 - c^2,\\
\mathfrak{F} &= \beta\gamma,\\
\mathfrak{G} &= \gamma\alpha,\\
\mathfrak{H} &= \alpha\beta;
\end{aligned}$$

and the equation of the system of diametral planes becomes
$$\begin{aligned}
\Theta = 0 = x^3.\alpha^2\beta\gamma\,(c^2-b^2) + y^3.\beta^2\gamma\alpha\,(a^2-c^2) + z^3.\gamma^2\alpha\beta\,(b^2-a^2)&\\
+ \gamma\alpha\,\{\alpha^2(c^2-b^2) + \beta^2(b^2+c^2-2a^2) - \gamma^2(b^2-a^2)&\\
+ (b^2-a^2)(c^2-b^2)\}\,&yz^2\\
+ \alpha\beta\,\{-\alpha^2(c^2-b^2) + \beta^2(a^2-c^2) + \gamma^2(c^2+a^2-2b^2)&\\
+ (c^2-b^2)(a^2-c^2)\}\,&zx^2\\
+ \gamma\alpha\,\{\alpha^2(a^2+b^2-2c^2) - \beta^2(a^2-c^2) + \gamma^2(b^2-a^2)&\\
+ (a^2-c^2)(b^2-a^2)\}\,&xy^2\\
- \alpha\beta\,\{\alpha^2(c^2-b^2) - \beta^2(a^2-c^2) - \gamma^2(b^2+c^2-2a^2)&\\
- (a^2-c^2)(c^2-b^2)\}\,&y^2z\\
- \beta\gamma\,\{-\alpha^2(c^2+a^2-2b^2) + \beta^2(a^2-c^2) - \gamma^2(b^2-a^2)&\\
- (b^2-a^2)(a^2-c^2)\}\,&z^2x\\
- \gamma\alpha\,\{-\alpha^2(c^2-b^2) - \beta^2(a^2+b^2-2c^2)&\\
+ \gamma^2(b^2-a^2) - (c^2-b^2)(b^2-a^2)\}\,&x^2y\\
+ \{(a^2-b^2)(b^2-c^2)(c^2-a^2)&\\
+(\alpha^4+\beta^2\gamma^2)(c^2-b^2)+(\beta^4+\gamma^2\alpha^2)(a^2-c^2)+(\gamma^4+\alpha^2\beta^2)(b^2-a^2)&\\
+ \alpha^2(b^2-c^2)(2a^2-b^2-c^2) + \beta^2(c^2-a^2)(2b^2-c^2-a^2)&\\
+ \gamma^2(a^2-b^2)(2c^2-a^2-b^2)\}\,&xyz.
\end{aligned}$$

And since this is a function of a^2-b^2, b^2-c^2, and c^2-a^2, the equation is the same for all confocal ellipsoids; whence the known theorem, "The axes of the circumscribing cone having its vertex in a given point P, are tangents to the curves of intersection of the three surfaces, confocal with the given surface, which pass through the point P."

D' après l'expression de V, on a

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=-4\pi f(x, y, z), \quad \text{ou} = 0,$$

suivant que le point (x, y, z) appartient ou non à la masse M.

Pour plus de simplicité, écrivons toujours

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=-4\pi f(x, y, z),$$

en regardant la fonction $f(x, y, z)$ comme nulle hors de la masse M; et combinons cette équation avec cette autre de forme analogue

$$\frac{d^2U}{dx^2}+\frac{d^2U}{dy^2}+\frac{d^2U}{dz^2}=0,$$

où nous supposons que U est une fonction de x, y, z, qui reste finie et continue ainsi que ses dérivées dans tout l'espace intérieur à (A). Nous aurons

$$V\frac{d^2U}{dx^2}-U\frac{d^2V}{dx^2}+V\frac{d^2U}{dy^2}-U\frac{d^2V}{dy^2}+V\frac{d^2U}{dz^2}-U\frac{d^2V}{dz^2}$$
$$=4\pi Uf(x, y, z).$$

Multiplions par $dx\,dy\,dz$, et intégrons dans tout l'espace intérieur à (A). En conservant à ds et à $d\omega$ la même signification que ci-dessus, on trouve, après des transformations bien connues:

$$\iint V\frac{dU}{ds}\,d\omega-\iint U\frac{dV}{ds}\,d\omega=4\pi\iiint Uf(x, y, z)\,dx\,dy\,dz.$$

Mais l'équation en U est satisfaite par $U=x$; nous avons donc:

$$\iint V\frac{dx}{ds}\,d\omega-\iint x\frac{dV}{ds}\,d\omega=4\pi\iiint xf(x, y, z)\,dx\,dy\,dz.$$

L' intégrale triple du second membre, divisée par M, donne l'abscisse du centre de gravité de la masse M. Ce centre étant à l' origine des coordonnées, l' intégrale dont nous parlons est nulle. Je vais prouver que l' intégrale $\iint V\frac{dx}{ds}\,d\omega$ l'est aussi. D' abord on peut faire sortir V du signe $\int$, puisque, sur la surface (A), V est constant. Observons ensuite que $\frac{dx}{ds}$ a pour valeur le cosinus de l'angle α que la normale ds fait avec l'axe des x. Notre intégrale

we deduce

$$\frac{1}{4\pi}\iint x^2 \frac{-dV}{ds}\,d\omega = \frac{1}{2\pi}\iiint (V-K)\,dx\,dy\,dz$$
$$+\iiint x^2 f(x,y,z)\,dx\,dy\,dz.$$

Let A, B, C be the moments of inertia of the mass M round the axes of coordinates, and A_1, B_1, C_1 those of the shell, round the same axes, it being supposed that the quantity of matter of the shell is the same as that of M;* the preceding equation, and the two others which correspond relatively to the axes of y and z, are with this notation,

$$A_1 = Q + A,\ B_1 = Q + B,\ C_1 = Q + C \ldots\ldots (2),\dagger$$

where $$Q, = \frac{1}{2\pi}\iiint (V-K)\,dx\,dy\,dz,$$

is a quantity which is independent of the position of the origin.

From equations (2), we have

$$B - C = B_1 - C_1,\ C - A = C_1 - A_1,\ A - B = A_1 - B_1 \ldots (3).$$

A demonstration of your theorem and of the theorems expressed by the equations (1) and (3) may be arrived at by comparing the expressions for the equal potentials ‡ produced by the mass M, and the shell at very distant points."

St. Peter's College, July 15, 1846.

ACTION OF A FORCE WHOSE DIRECTION ROTATES IN A PLANE.

By Andrew Bell.

This paper treats of the motion of a physical point acted on by a constant force whose direction passes through the point, and has a uniform angular motion in one plane.

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* In this case the "density" of the distribution at any point of the shell will be equal to $\frac{1}{4\pi}\cdot\frac{-dV}{ds}$. See Vol. III., p. 75.

† If the origin be taken at the centre of gravity, and the axes of coordinates principal axes of M, (and therefore of the shell, according to the proposition enunciated above,) these equations shew that the "central ellipsoid" (see note to p. 202) for the shell is confocal with that for the body M.

‡ A shell constructed round the mass M, in the manner described by M. Liouville, with a quantity of matter equal to M, exerts the same force upon points without the shell, as was proved first by Green, (see also Vol. III., p. 75); and, since the potential of each vanishes at an infinite distance, it follows that the two bodies produce equal potentials at every point without the shell.

If the point is at the origin when $t = 0$, then

$$x = \frac{\phi}{r^2}(1 - \cos rt) + v't, \quad y = \frac{\phi}{r^2}(rt - \sin rt) + v''t.$$

Let m, n be such numbers that $v' = \frac{m}{r}\phi$, $v'' = \frac{n}{r}\phi$, then

$$x = \frac{\phi}{r^2}(mrt + 1 - \cos rt), \quad y = \frac{\phi}{r^2}\{(1 + n)\,rt - \sin rt\}.$$

This system of equations is that of a species of oblique cycloid, or rather of a series of such cycloids, the line of whose bases passes through the origin, and is expressed by the equation

$$x = \frac{m\phi}{r}\,y = cy, \quad \text{if } c = \frac{m\phi}{r}.$$

Since $1 - \cos rt$ is never negative, the value of x is never less than that obtained from the equation $x = cy$, so that no part of the series of equal curves lies between the line of the bases and the axes of y. The points in which the curves meet the line of the bases, will be found by assuming $rt = 2e\pi$, where e is any integer; for then

$$x = \frac{m\phi}{r}\,t = ct,$$

which indicates a point of this line. The successive points of meeting are found by giving e the consecutive integral values 1, 2, 3, ... The corresponding values of y are

$$y = 2e\,(1 + n)\,\pi\,\frac{\phi}{r^2}.$$

The axis is the greatest value of the absciss of the curve reckoned from the base line, and is

$$x' = \frac{2\phi}{r^2},$$

and the length of the oblique base projected on the ordinate is

$$= 2\,(1 + n)\,\pi\,\frac{\phi}{r^2}.$$

The type of the curve is the common, the curtate, or prolate cycloid, according as n is zero, negative, or positive.

If when $t = 0$, $v' = 0$, or the point have merely an initial motion in the direction of the axis y, this motion is represented by the system

$$x = \frac{\phi}{r^2}(1 - \cos rt), \quad y = \frac{\phi}{r^2}\{(1 + n)\,rt - \sin rt\}$$

These equations are the general equation of the cycloid, and belong to a series of equal cycloidal curves whose bases lie in the axis of y, and whose axes are parallel to the axis of x.

The length of the axis is $\frac{2\phi}{r^2}$, and of the base $2\pi\,(1+n)\,\frac{\phi}{r^2}$.

It can easily be proved, by determining when $\frac{dy}{dx}$ is $= 0$, or $= \infty$, that according as n is zero, positive, or negative, the curves are common, prolate, or curtate cycloids.

Since the last equation becomes that of the common cycloid when $n = 0$, it appears that the general equation to the cycloid is deducible from that of the common cycloid by adding to the value of y the term $\frac{n}{r}\,\phi t$, which is proportional to the time or to θ; consequently this term indicates the impression, on a point moving in a common cycloid, of a uniform velocity, in the direction of the base, and $= \frac{n}{r}\,\phi t$, in addition to the component due to its oscillation. Hence a common cycloidal pendulum may be made to oscillate in a prolate or curtate cycloid by impressing on it and its cycloidal cheeks an initial uniform velocity in the direction of its base, according as the direction of this velocity is towards the side to which the pendulum is to move, or towards the opposite side; the cheeks being constrained to retain their uniform velocity.

The result of this investigation establishes the fact, that a constant and uniformly rotating force is capable of producing a progressive motion, and it also affords another remarkable physical property of the cycloid.

MATHEMATICAL NOTES.

I. Let $$A^3 - 3AB = D^3:$$

and, consequently, $B = -\frac{1}{3A}\,(D^3 - A^3)$;

then the equation at line 3, p. 249 of the second volume of the former series of this Journal becomes

$$y = -\frac{1}{3A}\cdot\frac{D^3 - A^3}{D - A}.$$

The further reductions are obvious. By writing aD for D (where a is one of the values of $1^{\frac{1}{3}}$) we obtain the three values of y.

J. C.

Devereux Court, March 17, 1846.

II. If
$$\begin{aligned} A &= aa' - bb' - cc', & D &= bc' + cb', \\ B &= bb' - cc' - aa', & E &= ca' + ac', \\ C &= cc' - aa' - bb', & F &= ab' + ba'; \end{aligned}$$
then
$$ABC - AD^2 - BE^2 - CF^2 + 2DEF =$$
$$(a^2 + b^2 + c^2)(aa' + bb' + cc')(a'^2 + b'^2 + c'^2).$$

Under the same conditions
$$(A + B)(B + C)(C + A) - 2DEF =$$
$$(A + B)F^2 + (B + C)D^2 + (C + A)E^2.$$

H. (1).

III. *On the Equation of Payments.*—In the tenth No. of the *Cambridge Mathematical Journal* the Theory of the Equation of Payments is briefly considered. The method there followed, in the case of simple interest, leads to the result ordinarily employed, as an approximate solution; whereas it is in fact as much entitled to be considered exact as any solution obtained on the supposition of simple interest can be. The reason of this misapprehension has been overlooked in the above-mentioned paper, as well as in the common treatises on Algebra.

Suppose that two sums, s_1 and s_2, are due at two periods, t_1 and t_2 respectively; that it is required to find the period T at which the sum $s_1 + s_2$ may be paid without injury to either party, simple interest being allowed.

There are three methods of making the arrangement, which at first sight appear equally fair.

1. The time T ought to be such, that the present worth of s_1 due at t_1, together with the present worth of s_2 due at t_2, shall be equal to the present worth of $s_1 + s_2$ due at T.

2. It ought to be such, that the interest of s_1 from the time t_1 to the time T, shall be equal to the discount of s_2 for the interval between T and t_2.

3. Or lastly, it ought to be such that s_1, with its interest from t_1 to t_2, together with s_2, shall be equal to $s_1 + s_2$ with its interest during the interval between T and t_2.

If these methods are equally just, they ought to give the same value of T. Let us see whether they do so.

at the time t_1, but upon the former only of these quantities is he to receive interest from the time t_1, *because the interest is simple:* whereas, if he receive s_1 at the time t_1, he will have the same sum at that time as before, but will receive interest *on the whole* of it from the time t_1.

The imaginary case then which we have substituted for the actual one, and which we have expressed in equation (1), is not equivalent to it, and we cannot therefore rely upon the result.

For the same reason the result in the second case is not correct. It requires no further explanation than to say, that discount implies the calculation of the present worth, and so introduces an error similar to that in the first method. But in the third case, the creditor will be in the same position at the time t_2, and therefore at any subsequent time, as if the two separate payments were made.

If r be supposed so small that the terms of which it is a factor may be neglected, equations (1) and (2) will agree with (3).

Since we have shewn the disagreement of our results in the three cases to arise from the fact of simple interest being considered, it ought to disappear when the interest is compound. We should thus have, for the first case,

$$\frac{s_1}{(1+r)^{t_1}} + \frac{s_2}{(1+r)^{t_2}} = \frac{s_1+s_2}{(1+r)^T},$$

therefore $$s_1(1+r)^{t_2-t_1} + s_2 = (s_1+s_2)(1+r)^{t_2-T} \;\ldots\; (4);$$

for the second,

$$s_1(1+r)^{T-t_1} - s_1 = s_2 - \frac{s_2}{(1+r)^{t_2-T}},$$

therefore $$s_1(1+r)^{t_2-t_1} + s_2 = (s_1+s_2)(1+r)^{t_2-T} \;\ldots\; (5);$$

and for the third,

$$s_1(1+r)^{t_2-t_1} + s_2 = (s_1+s_2)(1+r)^{t_2-T} \;\ldots\; (6);$$

and the three resulting equations now coincide.

H. Y.

END OF VOL. I.

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HEAT, ELECT

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THE CAMBRIDGE AND DUBLIN MATHEMATICAL JOURNAL.

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VOL. II.

(BEING VOL. VI. OF THE CAMBRIDGE MATHEMATICAL JOURNAL.)

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INDEX TO VOL. II.

GEOMETRY OF THREE DIMENSIONS.

ALGEBRA.

DIFFERENTIAL AND INTEGRAL CALCULUS.

MECHANICS.

MECHANICS—*continued*,

Notes and other passages enclosed in brackets [ ], are insertions made by the Editor.

The date of actual publication of any article, or portion of an article, may be found by referring to the first page of the sheet in which it is contained. The Author's private date is frequently given, at the end of an article.—ED.

ERRATA.

IN the memoir "On a Multiple Integral connected with the theory of Attractions," in the denominator of the value of U given by the formula (14), p. 221, *for* $\Gamma(\frac{1}{2}n-q)$ *read* $\Gamma(\frac{1}{2}n+q)$, and in the next line *for* $\Gamma(\frac{1}{2}n+q)$ *read* $\Gamma(\frac{1}{2}n-q)$.

then u will be a function of z and r. The transformed equation is easily found to be

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \frac{d^2u}{dz^2} = 0 \ldots\ldots\ldots\ldots (2),$$

which we proceed to discuss.

Writing the equation in the form

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} + r^2\frac{d^2u}{dz^2} = 0 \ldots\ldots\ldots (3),$$

let $r = \epsilon^\theta$, and let the symbol $\frac{d}{d\theta}$ be represented by D, then

$$r\frac{d}{dr} = D. \quad r^2\frac{d^2}{dr^2} = D(D-1);$$

$$\therefore \quad r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} = D^2u,$$

and the *symbolical* form of the differential equation is

$$D^2u + \frac{d^2}{dz^2}\epsilon^{2\theta}u = 0 \ldots\ldots\ldots\ldots (4).$$

The method which we shall employ in the solution of this equation, is that developed in the *Philosophical Transactions* for 1844, Part II. (On a General Method in Analysis), and partially explained in the first Number of this *Journal* (On Laplace's Equation). We shall first obtain the complete integral by series. We shall then deduce a particular solution, in the form of a definite integral, and shall examine the relation which it bears to the different parts of the general solution.

The equation $D^2u = 0$ would give

$$u = A + B\theta \ldots\ldots\ldots\ldots\ldots (5).$$

Substitute this value in (4); regarding A and B as variable parameters, we have

$$D^2A + \frac{d^2}{dz^2}\epsilon^{2\theta}A + 2DB + \left(D^2B + \frac{d^2}{dz^2}\epsilon^{2\theta}B\right)\theta = 0,$$

which affords the system of equations

$$D^2A + \frac{d^2}{dz^2}\epsilon^{2\theta}A + 2DB = 0,$$

$$D^2B + \frac{d^2}{dz^2}\epsilon^{2\theta}B = 0.$$

A being a series already determined, but reducible to a definite integral, viz.

$$A = \int_0^\pi d\theta\, \psi(z + r\cos\theta\sqrt{-1}) \ldots\ldots\ldots (20),$$

whenever the definite integral in (19) vanishes.

Now if we suppose the second member of (19) to represent the potential of a solid of revolution on an external point, it is necessary that the definite integral in the second term should be assumed to vanish; for otherwise the value of u would be infinite, were the attracted point in the axis of revolution, since $r = 0$ renders $\log(r)$ infinite. We have therefore

$$u = A = \int_0^\pi d\theta\, \psi(z + r\cos\theta\sqrt{-1}) \text{ by (20).}$$

Let $f(z)$ represent the potential on any exterior point z, in the axis of revolution, then

$$f(z) = \int_0^\pi d\theta\, \psi(z) = \pi\psi(z).$$

$$\therefore\quad \psi(z) = \frac{1}{\pi} f(z),$$

whence $$u = \frac{1}{\pi}\int_0^\pi d\theta f(z + r\cos\theta\sqrt{-1}) \ldots\ldots (21),$$

which is the expression required.

Some interesting consequences flow from this theorem. If the potential of a solid of revolution on every external point in the axis be constant, we have

$$f(z) = c,$$

$$u = \frac{1}{\pi}\int_0^\pi c\,d\theta = c.$$

Hence the potential on points out of the axis will be constant also, and the attraction will vanish. We have examples of this case in some closed shells and hollow cylinders of infinite length, as respects points situate on their hollow interiors. Points exterior to the outer surface are not continuous with the above, and require a separate determination of the arbitrary function. But if the surface is not closed, however small may be the aperture, or if the cylinder is of finite length, all points within the concave or without the convex surface are to be considered as included in one application of the general formula.

It would be interesting to verify the general theorem of this paper, by applying it to the case of a circular ring, and comparing the result with the one obtained by ordinary integration. I shall simply indicate the equation, the truth of which would for this purpose require independent proof.

[redacted]

as compared with the one which I have already given; but they throw an interesting light on the subject of Symbolical Algebra, and serve to illustrate some general doctrines in Analysis.

The equation which we shall consider is the following, viz.

$$\pi_m \pi_n u + q\rho u = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

in which u is the quantity to be determined, and the symbols π_m, π_n, ρ, applied to any subject u, combine according to the two laws

$$\pi_m \rho = \rho \pi_{m+1}, \quad \pi_n \rho = \rho \pi_{n+1} \ldots\ldots\ldots (2),$$

$$\pi_m \pi_n = \pi_n \pi_m + a(n - m)\rho \ldots\ldots\ldots\ldots (3).$$

The equations (2) are seen to be expressions of one law, π_m, π_n, differing only in the constants m and n. We suppose a to be an arbitrary constant.

Assume $u = \pi_{m+1} v$; we have by (1)

$$\pi_m \pi_n \pi_{m+1} v + q\rho \pi_{m+1} v = 0;$$

$$\therefore \quad \pi_m \pi_n \pi_{m+1} v + q \pi_m \rho v = 0 \quad \text{by (2)},$$

$$\pi_n \pi_{m+1} v + q\rho v = 0.^*$$

But $\pi_n \pi_{m+1} = \pi_{m+1} \pi_n + a(m + 1 - n)\rho$, by (3); therefore, on substitution,

$$\pi_{m+1} \pi_n v + \{q + a(m - n + 1)\} \rho v = 0.$$

Let $v = \pi_{m+2} w$; then, by inspection,

$$\pi_{m+2} \pi_n w + \{q + a(m - n + 1) + a(m - n + 2)\} \rho w = 0.$$

Continuing these transformations, it is evident that, if we suppose in the original equation

$$u = \pi_{m+1} \pi_{m+2} \ldots \pi_{m+r} v,$$

we shall have

$$\pi_{m+r} \pi_n v + \{q + a(m - n + 1) + a(m - n + 2) \ldots + a(m - n + r)\} \rho v = 0.$$

Or $$\pi_{m+r} \pi_n v + \left\{q + ar(m - n) + a\frac{r(r + 1)}{2}\right\} \rho v = 0.$$

If it then be possible by an integer value of r to satisfy the equation

$$q + ar(m - n) + a\frac{r(r + 1)}{2} = 0 \ldots\ldots\ldots (4),$$

we shall have $$\pi_{m+r} \pi_n v = 0,$$

$$v = \pi^{-1}_n \pi^{-1}_{m+r} 0;$$

---

* Strictly $\pi_n \pi_{m+1} v + q\rho v = \pi^{-1}_m 0$. The assumption in the text is lawful, if the result (5) gives the requisite number of arbitrary constants; a condition which is satisfied in the example adduced.

$$\therefore \quad u = \pi_{m+1}\pi_{m+2}\,.\,.\,\pi_{m+r}\pi_n^{-1}\pi^{-1}_{m+r}0 \,.\,.\,.\,.\,.\,.\,.\,.\,.\,. (5),$$

which is a complete solution of the equation proposed.

As the equation determining r has two roots, it may be inferred that there are two solutions, which may be denominated as conjugate to each other. The existence and character of the second solution will be most distinctly presented by the following analysis.

Resuming the equation

$$\pi_m\pi_n u + q\rho u = 0.$$

Let us suppose $u = \pi_m^{-1}v$, then

$$\pi_m\pi_n\pi_m^{-1}v + q\rho\pi_m^{-1}v = 0$$

$$\{\pi_n\pi_m + a(n-m)\,\rho\}\ \pi_m^{-1}v + q\rho\pi_m^{-1}v = 0, \text{ by (3)},$$

$$\pi_n v + \{q + a(n-m)\}\ \rho\pi_m^{-1}v = 0,$$

$$\pi_{m-1}\pi_n v + \{q + a(n-m)\}\ \pi_{m-1}\rho\pi_m^{-1}v = 0.$$

But, by (2), $\pi_{m-1}\rho = \rho\pi_m$. Substituting,

$$\pi_{m-1}\pi_n v + \{q + a(n-m)\}\ \rho v = 0.$$

Hence it is evident that the compound substitution

$$u = \pi_m^{-1}\pi^{-1}_{m-1}\,.\,.\,\pi^{-1}_{m-r+1}v \,.\,.\,.\,.\,.\,.\,.\,.\,.\,. (6),$$

would give

$$\pi_{m-r}\pi_n v + \{q + a(n-m)\,.\,.\,+ a(n-m+r-1)\}\ \rho v = 0\,.\,.(7).$$

Or $$\pi_{m-r}\pi_n v + \left\{q + a(n-m)\,r + a\,\frac{r(r-1)}{2}\right\}\rho v = 0.$$

Hence if we determine r by the equation

$$q + a(n-m)r + a\,\frac{r(r-1)}{2} = 0 \,.\,.\,.\,.\,.\,.\,.\,. (8),$$

we shall have
$$\pi_{m-r}\pi_n v = 0,$$

$$v = \pi_n^{-1}\pi^{-1}_{m-r}0,$$

$$u = \pi_m^{-1}\pi^{-1}_{m-1}\,.\,.\,\pi^{-1}_{m-r+1}\pi_n^{-1}\pi^{-1}_{m-r}0 \,.\,.\,.\,.\,.\,. (9);$$

so that the two conjugate solutions, exhibited at one view, are

$$\left.\begin{array}{l} u = \pi_{m+1}\pi_{m+2}\,.\,.\,\pi_{m+r}\pi_n^{-1}\pi^{-1}_{m+r}0 \\ u = \pi_m^{-1}\pi^{-1}_{m-1}\,.\,.\,\pi^{-1}_{m-r+1}\pi_n^{-1}\pi^{-1}_{m-r}0 \end{array}\right\} \,.\,.\,.\,.\,.\,. (10),$$

the values of r in the two cases being respectively determined by the equations

$$\left.\begin{array}{l} q + a(m-n)\,r + a\,\dfrac{r(r+1)}{2} = 0 \\[2ex] q + a(n-m)\,r + a\,\dfrac{r(r-1)}{2} = 0 \end{array}\right\} \,.\,.\,.\,. (11).$$

[redacted]

The roots of the one equation are evidently those of the other with changed signs. If both solutions are available, each equation will have a positive and a negative root, the former belonging to the solution with which it is connected, the latter with its sign changed to the conjugate solution.

It is an obvious corollary from the above, that if α and β be constants, then the solution of the equation

$$(\pi_m + \alpha)(\pi_n + \beta)\, u + q\rho u = 0 \;\dots\dots\dots\; (12),$$

will be be exhibited in either of the conjugate forms,

$$\left.\begin{aligned} u &= (\pi_{m+1} + \alpha)(\pi_{m+2} + \alpha)\dots(\pi_{m+r} + \alpha)\pi_n + \beta)^{-1}(\pi_{m+r} + \alpha)^{-1}\,0 \\ u &= (\pi_m + \alpha)^{-1}(\pi_{m-1} + \alpha)^{-1}\dots(\pi_{m-r+1} + \alpha)^{-1}(\pi_n + \beta)^{-1}(\pi_{m-r} + \beta)^{-1}\,0 \end{aligned}\right\}\dots(13),$$

the values of r being determined as before.

It remains to seek an interpretation of our symbols, and for this purpose let us assume

$$\pi_m = \phi(\mu)\frac{d}{d\mu} + m\phi'(\mu),\; \rho = \phi(\mu) \;\dots\dots\; (14).$$

Then
$$\begin{aligned} \pi_m\rho u &= \left\{\phi(\mu)\frac{d}{d\mu} + m\phi'(\mu)\right\}\phi(\mu)\, u, \\ &= \phi(\mu)^2\frac{du}{d\mu} + \phi(\mu)\,\phi'(\mu)\, u + m\phi(\mu)\,\phi'(\mu)\, u, \\ &= \phi(\mu)\left\{\phi(\mu)\frac{d}{d\mu} + (m+1)\,\phi'(\mu)\right\} u, \\ &= \rho\pi_{m+1}u \;\dots\dots\dots\dots\dots\dots\dots\dots\; (15). \end{aligned}$$

Secondly

$$\begin{aligned} \pi_m\pi_n u &= \left\{\phi(\mu)\frac{d}{d\mu} + m\phi'(\mu)\right\}\left\{\phi(\mu)\frac{d}{d\mu} + n\phi'(\mu)\right\} u, \\ &= \left\{\phi(\mu)\frac{d}{d\mu}\right\}^2 u + (m+n)\,\phi(\mu)\,\phi'(\mu)\frac{d}{d\mu} \\ &\qquad + n\phi(\mu)\,\phi''(\mu)\, u + mn\,\{\phi'(\mu)\}^2 u, \\ \pi_n\pi_m u &= \left\{\phi(\mu)\frac{d}{d\mu}\right\}^2 u + (m+n)\,\phi(\mu)\,\phi'(\mu)\frac{du}{d\mu} \\ &\qquad + m\phi(m)\,\phi''(\mu)\, u + mn\,\{\phi'(\mu)\}^2 u: \end{aligned}$$

therefore
$$\begin{aligned} (\pi_m\pi_n - \pi_n\pi_m)\, u &= (n-m)\,\phi(\mu)\,\phi''(\mu)\, u, \\ &= (n-m)\,\rho\phi''(\mu)\, u. \end{aligned}$$

Or, dropping the subject, u

$$\pi_m\pi_n = \pi_n\pi_m + (n-m)\,\rho\phi''(\mu)\, u \;\dots\dots\; (16);$$

and that this may be identical with (3), we must have

$$\phi''(\mu) = a)$$

We may remark that the process of reduction might have been so ordered as to have eliminated the last operating factor in each solution. This would have detracted from the generality of the first solution, in which there is but one other inverse factor, but not of the second. We have therefore for the sake of symmetry retained the factor in both. To shew how it might have been evaded, let us resume the equations (6) and (7), and writing the second in the form

$$\pi_n\pi_{m-r}v + \{q + a(n-m)\ldots + a(n-m+r-1) + a(n-m+r)\}\rho v = 0,$$

or $$\pi_n\pi_{m-r}v + \left\{q + a(n-m)(r+1) + a\frac{r(r+1)}{2}\right\} = 0,$$

we have $$\pi_n\pi_{m-r}v = 0, \quad \text{or} \quad v = \pi_{m-r}^{-1}\pi_n^{-1}0,$$

if $$q + a(n-m)(r+1) + a\frac{r(r+1)}{2} = 0.$$

Hence $u = \pi_m^{-1}\pi_{m-1}^{-1}\ldots\pi_{m-r+1}^{-1}\pi_{m-r}^{-1}\pi_n^{-1}0$, or writing $r-1$, for r

$$u = \pi_m^{-1}\pi_{m-1}^{-1}\ldots\pi_{m-r+1}\,\pi_n^{-1}0 \ldots\ldots\ldots\ldots(21),$$

if $$q + a(n-m)r + a\frac{r(r-1)}{2} = 0,$$

which differs only from the solution before obtained by the last factor.

The direct operations implied in the above solutions will involve differentiation, and the inverse ones the solution of a partial differential equation of the first order. We shall not exhibit the results, as it is clear that neither of the solutions can be freed from integral signs, but shall only remark that the second solution, freed as above from its last factor, is equivalent to the result obtained by Mr. Hargreave in the *Philosophical Transactions*.

The investigation we have entered upon is chiefly valuable, as presenting to us what will be thought a very curious chapter in symbolical algebra, and introducing us to the family of which Laplace's equation is a member. But it must be confessed that they are an interesting rather than an amiable group.

To give completeness to my former paper, I ought to have illustrated the general value of P_n deduced from the integral by actually calculating a few coefficients. This is an extremely simple matter, as all the operations are direct. It must be remembered that the product $1.2.\ldots.p$ like $\Gamma(p+1)$ becomes 1 when $p = 0$.

Lincoln, Aug. 18, 1846.

[redacted]

The diameter conjugate to (2) is $\frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1}$, and as this line is the intersection of the diametral planes parallel to (3) and (4), we must have

$$\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} + \frac{z_1z_2}{c^2} = 0 \;\dots\dots\dots\dots (5),$$

and

$$\frac{x_1x_3}{a^2} + \frac{y_1y_3}{b^2} + \frac{z_1z_3}{c^2} = 0 \;\dots\dots\dots\dots (6).$$

By similarly considering the diameters conjugate to (3) or (4), we shall get one of the equations just deduced, together with the following,

$$\frac{x_2x_3}{a^2} + \frac{y_2y_3}{b^2} + \frac{z_2z_3}{c^2} = 0 \;\dots\dots\dots\dots (7).$$

Moreover the conjugate points being on the surface of the ellipsoid, we have

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1 \;\dots\dots\dots\dots (8),$$

$$\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} + \frac{z_2^2}{c^2} = 1 \;\dots\dots\dots\dots (9),$$

$$\frac{x_3^2}{a^2} + \frac{y_3^2}{b^2} + \frac{z_3^2}{c^2} = 1 \;\dots\dots\dots\dots (10).$$

Now, if in (A) we substitute $\frac{x_1}{a}$, $\frac{y_1}{b}$, $\frac{z_1}{c}$, $\frac{x_2}{a}$, &c. for l_1, m_1, n_1, l_2, &c., we shall get (8, 9, 10, 5, 6, 7), hence, as the results of this substitution are true, the equations (B) and (C) will still be true after undergoing the same transformation. We shall thus, after an obvious reduction, get the following equations,

$$x_1^2 + x_2^2 + x_3^2 = a^2 \;\dots\dots\dots\dots (11),$$

$$y_1^2 + y_2^2 + y_3^2 = b^2 \;\dots\dots\dots\dots (12),$$

$$z_1^2 + z_2^2 + z_3^2 = c^2 \;\dots\dots\dots\dots (13),$$

$$x_1y_1 + x_2y_2 + x_3y_3 = 0 \;\dots\dots\dots\dots (14),$$

$$x_1z_1 + x_2z_2 + x_3z_3 = 0 \;\dots\dots\dots\dots (15),$$

$$y_1z_1 + y_2z_2 + y_3z_3 = 0 \;\dots\dots\dots\dots (16).$$

$$\pm\frac{x_1}{a} = \frac{y_2z_3 - y_3z_2}{bc} \qquad \pm\frac{x_2}{a} = \frac{y_3z_1 - y_1z_3}{bc} \qquad \pm\frac{x_3}{a} = \frac{y_1z_2 - y_2z_1}{bc} \;\dots(17).$$

$$\pm\frac{y_1}{b} = \frac{x_3z_2 - x_2z_3}{ac} \qquad \pm\frac{y_2}{b} = \frac{x_1z_3 - x_3z_1}{ac} \qquad \pm\frac{y_3}{b} = \frac{x_2z_1 - x_1z_2}{ac} \;\dots(18).$$

$$\pm\frac{z_1}{c} = \frac{x_2y_3 - x_3y_2}{ab} \qquad \pm\frac{z_2}{c} = \frac{x_3y_1 - x_1y_3}{ab} \qquad x\,y_1 \;\dots(19).$$

of a parallelepiped of which three contiguous edges meet at the origin and terminate in the three points $(x_1y_1z_1)$, $(x_2y_2z_2)$, $(x_3y_3z_3)$, is

$$V = (x_2y_3 - x_3y_2)\,z_1 + (x_3y_1 - x_1y_3)\,z_2 + (x_1y_2 - x_2y_1)\,z_3$$

$$= (19)\,\frac{ab}{c}\,\{z_1^2 + z_2^2 + z_3^2\} = (13)\,abc \,\ldots\ldots\ldots\,(21).$$

Now the volume of the conjugate parallelepiped of which the conjugate tangent planes (2, 3, 4) are adjacent faces, is evidently eight times that just found; hence, we infer that

(E) Each conjugate parallelepiped circumscribing an ellipsoid, is equal to that constructed on the principal diameters.

Let $\widehat{r_1r_2}$ denote the angle included between the radii r_1 and r_2. The area of the parallelogram of which r_1 and r_2 are contiguous sides is $r_1r_2 \sin \widehat{r_1r_2}$, and the projections of this area on the planes of yz, zx, and xy, are $y_1z_2 - y_2z_1$, $x_2z_1 - x_1z_2$, and $x_1y_2 - x_2y_1$; hence, by the theory of projections (Gregory's *Solid Geom.* p. 14),

$$r_1^2r_2^2 \sin^2 \widehat{r_1r_2} = (y_1z_2 - y_2z_1)^2 + (x_2z_1 - x_1z_2)^2 + (x_1y_2 - x_2y_1)^2;$$

or reducing, by means of (17, 18, 19),

$$r_1^2r_2^2 \sin^2 \widehat{r_1r_2} = a^2b^2c^2 \left\{\frac{x_3^2}{a^4} + \frac{y_3^2}{b^4} + \frac{z_3^2}{c^4}\right\}$$

Similarly $$r_1^2r_3^2 \sin^2 \widehat{r_1r_3} = a^2b^2c^2 \left\{\frac{x_2^2}{a^4} + \frac{y_2^2}{b^4} + \frac{z_2^2}{c^4}\right\} \ldots.(22).$$

and $$r_2^2r_3^2 \sin^2 \widehat{r_2r_3} = a^2b^2c^2 \left\{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4}\right\}.$$

Add these and reduce by (11, 12, 13),

$$r_1^2r_2^2 \sin^2 \widehat{r_1r_2} + r_1^2r_3^2 \sin^2 \widehat{r_1r_3} + r_2^2r_3^2 \sin^2 \widehat{r_2r_3} = b^2c^2 + a^2c^2 + a^2b^2 \ldots(23),$$

which amounts to another well-known theorem; namely

(F) The sum of the squares of the parallelograms formed by each pair of conjugate diameters is equal to the sum of the squares of the rectangles under each pair of the principal diameters. Or, the sum of the squares of the faces of any conjugate parallelepiped is equal to the sum of the squares of the faces of the parallelepiped described on the principal diameters.

If p_1, p_2, and p_3 be the perpendiculars from the centre on the three conjugate tangent planes (2, 3, 4), we shall have

and this quotient equated to zero is the result of the elimination freed from extraneous factors. It only remains to demonstrate the formulæ (A), (B), and (C). Suppose in general that (k) denotes the sum of all the terms of the form $m^a n^b \dots$, which can be formed with a given combination of k letters out of the ϕ letters $m, n, p \dots$ And let $\Sigma\,(k)$ denote the sum of all the series (k) obtained by taking all the possible different combinations of k letters. It is evident that $\Sigma\,(k)$ is a multiple of (ϕ), [(ϕ) denoting of course the sum of all the terms $m^a n^b \dots$, $m, n \dots$ being any letters whatever out of the series $m, n, p \dots$]. Let g be the number of exponents $a, b, \dots$, then (ϕ) contains $[\phi]^g$ terms, also (k) contains $[k]^g$ terms, and the number of terms such as (k) in the sum $\Sigma\,(k)$ is $[\phi]^{\phi-k} \div [\phi - k]^{\phi-k}$. Whence evidently

$$\Sigma\,(k) = \frac{[\phi - g]^{\phi-k}}{[\phi - k]^{\phi-k}}\,(\phi).$$

Or, what comes to the same thing,

$$\Sigma\,(\phi - k) = \frac{[\phi - g]^k}{[k]^k}\,(\phi).$$

Let A be an indeterminate coefficient, σ a summatory sign referring to different systems of exponents; then

$$\therefore\quad \Sigma\sigma A\,(\phi - k) = \sigma\,\frac{[\phi - g]^k}{[k]^k}\,A\,(\phi).$$

Or, giving to k the values $1, 2 \dots \phi$, multiplying each equation by an arbitrary coefficient, and adding, putting also for shortness $\sigma A\,(\phi - k) = U_{\phi-k}$, we have

$$a_\phi U_\phi + a_{\phi-1}\Sigma U_{\phi-1} + \dots = \sigma\left(a_\phi + a_{\phi-1}\,.\,\frac{[\phi - g]^1}{[1]^1} + \dots\right) A\,(\phi);$$

whence in particular,

$$U_\phi - \Sigma U_{\phi-1} + \dots = \sigma\,\{0^{\phi-g} A\,(\phi)\},$$

$$\Sigma U_{\phi-1} - 2\Sigma U_{\phi-2} + \dots = \sigma\,\{(\phi - g)\,0^{\phi-g-1} A.(\phi)\},$$

which are still equations of considerable generality. If now $\phi = \theta$ and U_θ is a function of $m + n + p + \dots$ of the order θ, the quantity $\sigma\,\{0^{\theta-g} A\,(\theta)\}$ reduces itself to the single term of U_θ which contains the product $mnp \dots$ Hence, if

$$U_\theta = [a + m + n + p \dots, \theta]$$

in which afterwards $a = r - m - n - p - \dots$ we have the formula (A). Again, if $\phi = \theta + 1$, and $U_{\theta+1}$ a function of $m + n + p \dots$ of the order θ, the sum $\sigma\,\{0^{\theta+1-g} A\,.\,(\phi)\}$ vanishes; whence writing $U_{\theta+1} = [m + n + p \dots - 0, \theta]$, we have the formula ($B$).

α, β, γ the projections on three rectangular axes of coordinates, of the infinitely small displacement of a point (x, y, z) of the solid, it follows from Mr. Stokes' results that the equations of equilibrium, when the body is acted on by no forces except at its bounding surfaces, may be written as follows:

$$\left.\begin{aligned} -\frac{dp}{dx}+\frac{d^2\alpha}{dx^2}+\frac{d^2\alpha}{dy^2}+\frac{d^2\alpha}{dz^2}&=0\\ -\frac{dp}{dy}+\frac{d^2\beta}{dx^2}+\frac{d^2\beta}{dy^2}+\frac{d^2\beta}{dz^2}&=0\\ -\frac{dp}{dz}+\frac{d^2\gamma}{dx^2}+\frac{d^2\gamma}{dy^2}+\frac{d^2\gamma}{dz^2}&=0\\ p=-k\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)& \end{aligned}\right\}\ldots\ldots\ldots(1).$$

In the ideal limiting case in which the solid is incompressible, k will have an infinite value, and we shall have the relation

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0\ \ldots\ldots\ldots\ldots\ (2).$$

Hence equations (1) and (2) express the conditions of the interior equilibrium of an incompressible elastic solid. These equations are to be employed for the representation of the forces in the several physical problems considered in this paper.

Now equations (1) merely shew that the expression

$$\nabla^2\alpha.dx+\nabla^2\beta.dy+\nabla^2\gamma.dz\ \ldots\ldots\ldots\ldots(a),$$

$\left(\text{in which } \nabla^2 \text{ denotes the operation } \frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)$, must be a complete differential, and therefore any expressions for α, β, γ subject to this condition, which satisfy (2), will represent an interior state of the body which can be produced by the action of forces at its bounding surface or surfaces.

We may obtain a particular solution by assuming $\alpha dx + \beta dy + \gamma dz$ to be a complete differential. Again, if we suppose this expression not to be a complete differential, we may assume

$$\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)dx+\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\right)dy+\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right)dz\ldots(c)$$

to be a complete differential and find another solution; or lastly we may obtain a particular solution by means of a third supposition, according to which neither of these expressions is a complete differential. These three solutions I shall now proceed to consider, with reference to the representation of Electrical, Magnetic, and Galvanic forces.

components round lines parallel to the axes, of the infinitely small rotation which an element of the solid receives, besides its change of form, when $\alpha dx + \beta dy + \gamma dz$ is not a complete differential. This rotation therefore represents the resultant magnetic force, in direction and magnitude.

III.—*Galvanic Forces.*

Let $\nabla^2\alpha . dx + \nabla^2\beta . dy + \nabla^2\gamma . dz = - d\,\dfrac{lx + my + nz}{r^3}$,

which is true if

$$\left.\begin{aligned} \alpha &= \tfrac{1}{2}\frac{d}{dx}\frac{lx+my+nz}{r} - \frac{l}{r} \\ \beta &= \tfrac{1}{2}\frac{d}{dy}\frac{lx+my+nz}{r} - \frac{m}{r} \\ \gamma &= \tfrac{1}{2}\frac{d}{dz}\frac{lx+my+nz}{r} - \frac{n}{r} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (4).$$

It is readily verified that these expressions also satisfy equation (2), and hence they represent an interior state of the body which may be produced by externally applied forces. Now, we find by means of these equations

$$\left.\begin{aligned} \frac{d\beta}{dz} - \frac{d\gamma}{dy} &= \frac{mz - ny}{r^3} \\ \frac{d\gamma}{dx} - \frac{d\alpha}{dz} &= \frac{nx - lz}{r^3} \\ \frac{d\alpha}{dy} - \frac{d\beta}{dx} &= \frac{ly - mx}{r^3} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (\text{III.})$$

which are the expressions for the components of the force an infinitely small element of a galvanic current, in the direction l, m, n, at the origin, produces on a unit of magnetism at the point (x, y, z); the intensity of the current, multiplied by the length of the element, being unity. Thus we conclude that the rotation of any element of the solid, in the state expressed by (4), represents in direction and magnitude, the force of an element of a galvanic wire.

I should exceed my present limits were I to enter into a special examination of the states of a solid body representing various problems in electricity, magnetism, and galvanism, which must therefore be reserved for a future paper.

Glasgow College, *Nov.* 28, 1846.

how this degree is diminished if the given surface have multiple points or lines.

4. The degree of the reciprocal surface is plainly the same as the number of tangent planes which can be drawn to the surface through a given line: now we know that all the points of contact of tangent lines passing through a given point lie on a surface of the $n-1^{st}$ degree, which we call the $(n-1)^{st}$ polar surface of that point.

In order, then, to find the points of contact of planes passing through a fixed line, we have only to take the polar surfaces of any two points on this line. The intersections of these surfaces with the given one are the points of contact required; and since three surfaces respectively of the k^{th}, l^{th}, and m^{th} degrees intersect each other in $k.l.m$ points, the number of intersections in the present case will be $m.(m-1)^2$. This therefore is the degree of the surface reciprocal to one of the m^{th} degree.

5. There is another method by which we might have determined the degree of the reciprocal surface, the result of which does not at first sight appear the same as the preceding.

The degree of the reciprocal surface is the same as that of any plane section of it; but any plane section of the reciprocal surface is reciprocal to a tangent cone of the given surface. Now the degree of the cone touching a surface of the m^{th} degree is $m.m-1$, therefore (as the reciprocals of cones follow the same rules as curves) if the cone have no multiple sides, its reciprocal will be of the degree $m.(m-1)\{m.(m-1)-1\}$. We appear, then, to have arrived at a result contradictory to that of the preceding section.

6. I proceed to remove the apparent contradiction by establishing the following theorems. (1) "Every cone touching a surface of the m^{th} degree must in general have $\frac{m.(m-1)^2.m-2}{2}$ double sides, real or imaginary." (2) "Of these double sides, $m.m-1.m-2$ are cuspidal lines, and consequently there are only $\frac{m.m-1.m-2.m-3}{2}$ ordinary double lines." Assuming for a moment the truth of these theorems, we see that the degree of the curve reciprocal to this cone will be

$$m.m-1.\left\{m^2-m-1-2.\frac{m-2.m-3}{2}-3.(m-2)\right\}=m.(m-1)^2,$$

Its equation must be of the form $A^{-1} + B^{-1} + C^{-1} + D^{-1} = 0$, $A = 0$, &c. being the equations of the four sides of the pyramid formed by the double points. This belongs to the class of surfaces $A^m + B^m + C^m + D^m = 0$, whose reciprocal is of the same form, the new m being equal to $\dfrac{m}{m-1}$. In the present case the reciprocal is of the form $A^{\frac{1}{2}} + B^{\frac{1}{2}} + C^{\frac{1}{2}} + D^{\frac{1}{2}} = 0$, a surface of the fourth degree as we expected.

14. We found in curves that though the degree of the reciprocal is only reduced by two for an ordinary double point, nodal or conjugate, yet if the tangents at it coincide, the degree will be reduced by *three*. We should expect to find an analogous result for surfaces, and accordingly I find that such is the case when the tangent cone at the double point reduces itself to two planes, real or imaginary. In this case the two surfaces of the $n - 1^{st}$ degree, whose intersections with the original we employed to determine the degree of the reciprocal, will not only pass through the double point, but will also both touch the line of intersection of the two planes; hence it appears that the degree of the reciprocal will be diminished by 3.

15. An instance of such points we have in the surface of the third degree $A.B.C = D^3$, $A = 0$, &c. being the equations of planes. Here the three points ABD, ACD, BCD are double points, and the tangent cone at any reduces to two of the planes $A = 0$, $B = 0$, $C = 0$. But the reciprocal of this surface is another of the same form, reducing from the twelfth degree to the third, on account of the three double points just mentioned.

16. Again, if the two planes coincide, both the surfaces of the $n - 1^{st}$ degree must touch this plane, and it is not difficult to see that the degree of the reciprocal surface will be reduced by 6.

Multiple points of higher degrees present no difficulty.

17. The case of multiple *lines*, however, involves much more complicated considerations. Suppose a surface to have a double line, the two polar surfaces of the $n - 1^{st}$ degree will each pass through it, and the question becomes "In how many other points will three surfaces intersect which each pass through a given curve, that curve being a double line on one of them?"

Let us commence by the simpler question: "Three surfaces respectively of the m^{th}, n^{th}, and p^{th} degrees pass through a

which affect the degree of the reciprocal, and which we have not yet taken into consideration.

20. If a surface have a double line of any kind, in general at any point of it two planes can be drawn tangent to the surface; but there will always be a determinate number of points (which I call cuspidal points), at which the two tangent planes coincide, and for each of these points the degree of the reciprocal will be further diminished by one. Take the case where the double line is a right line (suppose the axis of z), the equation of the surface will be

$$Ay^2 + Byx + Cx^2 = 0,$$

A, B, C being any functions of the variables.

The tangent planes at any point are determined by the equation $A'y^2 + B'xy + C'x^2 = 0$, where $A'B'C'$ are the values which ABC take for that point, and it is evident that these planes will coincide at the points where the axis meets the surface $B^2 = 4AC$. This surface being of the $2m - 4^{\text{th}}$ degree, we are to add this number to the number $5m - 8$ already determined, and we find that if a surface have a double right line, the degree of its reciprocal is diminished $7m - 12$. Hence the reciprocal of a surface of the third degree which has a double line, is of the third degree, since this double line is necessarily a right line.

21. If the surface have a triple right line, proceeding by a precisely similar method, we find that the degree of the reciprocal surface is diminished by $20m - 48$.

And in general, that if a surface have a multiple right line of the degree r, we find, by the same method, that the reciprocal is diminished by $(r - 1).(3r + 1)\,m - 2r.(r^2 - 1)$. Hence if a surface have a multiple line of the degree $m - 1$ (which must be a right line, since no plane can cut it in more points than one), the degree of the reciprocal will be the same as that of the given surface.

22. Let us now suppose the double line to be the curve of intersection of a surface of the k^{th} with one of the l^{th} degree.

First let us consider if three surfaces pass through such a curve, in how many other points will they intersect. Take the case where the first surface is one of the k^{th} and one of $(m - k)^{\text{th}}$; the second, one of l^{th} and of $n - l^{\text{th}}$ degree; the third, one of the p^{th}; the number of points of intersection will be

$$k.(n - l).(p - l) + l.(m - k).(p - k) + p.(m - k).(n - l),$$

therefore the general number will in this case be diminished

and the tangent plane the plane of xy, the part of the equation below the third degree will be

$$Ay^2 + Bxy + Cx^2 + z(Dy + Ex + Fz + G) = 0.$$

The equation $Ay^2 + Bxy + Cx^2 = 0$ determines the directions of the tangents at the origin to the intersection of the surface with the tangent plane; and when these directions coincide, the origin is a parabolic point. If the equation be

$$(\alpha y + \beta x)^2 + z(Dy + Ex + Fz + G) = 0,$$

it is evident that the polar of the second degree will be a cone whose vertex is the intersection of the three planes $z = 0$, $\alpha y + \beta x = 0$, $Dy + Ex + Fz + G = 0$. Hence to find the parabolic points we have only to seek the locus of a point whose polar of the second degree shall be a cone; and the intersection of this locus with the given surface will determine the points in question. But the condition that the general equation of the second degree should represent a cone, is of the fourth degree with regard to the coefficients; hence the locus required will be of the $4.n - 2$ degree.

Trinity College, Dublin, Dec. 14, 1846.

INVESTIGATION OF THE VALUE OF $\int_0^\infty \frac{\sin x dx}{x}$.*

By Francis W. Newman, Professor of Latin in University College, London.

Put $A_n = \int_0^{n\pi} \frac{\sin x dx}{x}$. Then, observing that

$$\int_{n\pi}^{(n+1)\pi} \frac{\sin x dx}{x} = \int_0^\pi \frac{\cos n\pi \sin x' dx'}{n\pi + x'},$$

which vanishes when n = an infinite integer, since the denominator is then infinite; it follows, *a fortiori*, that

$$\int_{n\pi}^{n\pi+\mu} \frac{\sin x dx}{x}$$

vanishes when n = an infinite integer and μ is between 0 and π: consequently we find the value of A from A_n by supposing $n = \infty$, without any difference in the result, whether n be integer or fractional.

[* Before I received this article, Mr. Cayley had communicated to me a paper containing the application of the method here given to the evaluation of various integrals, both single and double; but his results have not yet been published. The solution given in the article published at present was given by Mr. Cayley as an example.—Ed.]

Now $\int_0^{n\pi} = \int_0^{\pi} + \int_{\pi}^{2\pi} + \int_{2\pi}^{3\pi} + \ldots + \int_{(n-1)\pi}^{n\pi}$.

Hence we have

$$A_n = \int_0^{\pi} \frac{\sin y_1 dy_1}{y_1} + \int_{\pi}^{2\pi} \frac{\sin y_2 dy_2}{y_2} + \ldots + \int_{(n-1)\pi}^{n\pi} \frac{\sin y_n dy_n}{y_n}:$$

since, when the limits are fixed, we may arbitrarily change x into y.

Let
$$y_1 = \pi - x;\quad y_2 = \pi + x;$$
$$y_3 = 3\pi - x;\quad y_4 = 3\pi + x;$$
$$y_5 = 5\pi - x;\quad y_6 = 5\pi + x;\ \&c\ldots$$

Then when $y_1, y_3, y_5, y_7 \ldots$ are at their lower limits, x is π; but at their upper limits, $x = 0$. The cases are reversed for $y_2, y_4, y_6, y_8 \ldots$. Observing, then, that $\int_{\pi}^{0} - du = \int_0^{\pi} du$, we change every term of the series into an integral, in which x varies between the same limits 0 and π; so that

$$A_n = \int_0^{\pi} \left\{ \frac{\sin x dx}{\pi - x} - \frac{\sin x dx}{\pi + x} + \frac{\sin x dx}{3\pi - x} - \frac{\sin x dx}{3\pi + x} + \frac{\sin x dx}{5\pi - x} - \&c. \ldots \text{to } n \text{ terms} \right\}.$$

Since the denominators increase, the fractions diminish; and as they are alternately positive and negative, the sum will approximate to a single real and finite limit, when $n = \infty$, by a well-known law of infinite series. Hence, making $n = \infty$, we get

$$A = \int_0^{\pi} \left\{ \frac{1}{\pi - x} - \frac{1}{\pi + x} + \frac{1}{3\pi - x} - \frac{1}{3\pi + x} + \&c.\ \&c. \right\} \sin x dx.$$

But by an easy result of the equation

$$\cos \frac{x}{2} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\left(1 - \frac{x^2}{25\pi^2}\right) \&c.\ \&c.$$

it is well known that the series within brackets $= \frac{1}{2} \tan \frac{x}{2}$;

$$\therefore\quad A = \int_0^{\pi} \tfrac{1}{2} \tan \frac{x}{2} \sin x dx = \int_0^{\pi} \sin^2 \frac{x}{2} dx = \int_0^{\pi} \frac{1 - \cos x}{2} dx$$

$$= \left(\frac{x}{2} - \frac{\sin x}{2}\right) \text{ within proper limits; or } A = \frac{\pi}{2}.$$

P.S. I ventured to send this Article to the Editor, at the request of a mathematician, who had doubted of the truth of the *result*, in consequence of the defectiveness of the current proof.—F. N.

ON LOGARITHMIC INTEGRALS OF THE SECOND ORDER.

By Francis W. Newman.

§. I.

1. The general formula $\int F_1x \,.\, \log F_2x \,.\, dx$, where $F_1 F_2$ denote rational functions, contains a variety of integrals, all of which, it will be shewn, can be reduced to *three*.

By the common method of finding $\int F_1x \,.\, dx$, we perceive that there is some rational function F_3 which fulfils the equation

$$F_1x = \frac{d}{dx} F_3x + \Sigma \frac{A}{x-e} + \Sigma \frac{px+q}{(x-\mu)^2+\nu^2}.$$

Also, if F_2x be reduced to the form of a single algebraic fraction, it may be denoted by $F'x \div F''x$, where F' and F'' are each *integer*. Consequently we may write

$$\log F_2x = \Sigma . A_1 \log (ax+b) + \Sigma . A_2 \log (a'x^2 + b'x + c').$$

It immediately follows that $\int F_1x \,.\, \log F_2x dx$ is separable into the two forms $\int F_1x . \log (ax+b)\, dx$ and $\int F_1x . \log (a'x^2+b'x+c')\, dx$. In the former, introduce the preceding value of F_1x, and we obtain for the integral

$$\log (ax+b) \,.\, F_3x - \int \frac{F_3x.adx}{ax+b}$$
$$+ \Sigma . A \int \frac{\log (ax+b)}{x-e} dx + \Sigma \int \log (ax+b) \,.\, \frac{(px+q)\, dx}{(x-\mu)^2+\nu^2}.$$

Of the three integrals which here appear, the first is rational. In the second assume $ax + b = mx'$; $\therefore\ a(x-e) = mx' - b - ae$. Assume farther, $m = b + ae$; then

$$\int \frac{\log (ax+b)\, dx}{x-e} = \log m . \int \frac{dx}{x-e} + \int \frac{\log x' . dx'}{x'-1},$$

provided that m, or $(b+ae)$, is positive. If otherwise, put $x = e + m'x''$, and $am' = -(b+ae)$;

$$\therefore \int \frac{\log (ax+b)\, dx}{x-e} = \log (ax+b) \,.\, \log \frac{x-e}{m'} - \int \frac{\log x'' . dx''}{x''-1}.$$

In either case we arrive at the elementary form

$$L(x) = \int_1 \frac{\log x.dx}{x-1} \ldots\ldots\ldots\ldots (1),$$

which Spence has tabulated. As for the integral

$$\int \log (ax+b) \,.\, \frac{(px+q)\, dx}{(x-\mu)^2+\nu^2},$$

[redacted]

the same assumption, $ax + b = mx'$, if we give to m a suitable constant value, produces two general forms which may be denoted by

$$\int \frac{\log x\,dx}{X} \text{ and } \tfrac{1}{2}\int \log x\, d\log X; \text{ if } X = x^2 - 2x\cos a + 1.$$

Let ω be an arc such that $\tan\omega = x\sin a \div (1 - x\cos a)$, or, what is the same, $x = \sin\omega \div \sin(\omega + a)$: then

$$d\omega = \frac{\sin a.dx}{X}; \text{ and } \sin a.\int\frac{\log x\,dx}{X}$$
$$= \int \log\sin\omega d\omega - \int \log\sin(\omega + a)\,d\omega.$$

Suppose ϟ to be a symbol for a new function, such that

$$\text{ϟ}(\omega) = -\int_0 \log\sin\omega.d\omega \ldots\ldots\ldots\ldots(2);$$

then
$$\sin a.\int_0 \frac{\log x\,dx}{X} = \text{ϟ}(\omega + a) - \text{ϟ}\omega - \text{ϟ}a \ldots.(2).$$

No similar reduction occurs, by which we can exterminate the arbitrary constant from the next integral; and we must be satisfied with writing

$$\Lambda(x, a) \text{ for } \int_0 \frac{\log x.(x - \cos a)\,dx}{x^2 - 2x\cos a + 1} \text{ or } \tfrac{1}{2}\int lx.dlX \ldots(3).$$

It will be sometimes convenient to put

$$\lambda(x, a) \text{ for } \tfrac{1}{2}\int_0 \log(x^2 - 2x\cos a + 1).\frac{dx}{x}\ldots.(4),$$

which is a supplemental function to Λ, and so related that

$$\Lambda(x, a) + \lambda(x, a) = \tfrac{1}{2}\log x.\log X.$$

We may write Λx, λx when no change of a is contemplated.

2. We have now to go back to $\int F_1 x.\log(ax^2 + bx + c)\,dx$. By substituting as before for F_1, we reduce the integral to

$$F_2 x.\log(ax^2 + bx + c) - \int F_2 x.\frac{(2ax + b)\,dx}{ax^2 + bx + c}$$
$$+ \Sigma A\int\frac{\log(ax^2 + bx + c)}{x - e}\,dx + \Sigma\int\log(ax^2 + bx + c).\frac{(px + q)\,dx}{(x - \mu)^2 + \nu^2}.$$

Of the three integrals remaining, the first is rational. The second is readily reduced to the form λ, by making $(x - e) = mx'$. The third, by making $x - \mu = mx'$, and determining m aright, produces the two new forms

$$X_1 = \int \log X.\frac{n\,dx}{x^2 + n^2}; \quad X_2 = \int \log X.\frac{x\,dx}{x^2 + n^2};$$

each of which has two arbitrary constants, a and n. But fortunately we can reduce X_1 to $\daleth$, and X_2 to L or λ. First, for X_1, put $x = n \tan \omega$, $n = \tan \nu$; $\dfrac{ndx}{x^2 + n^2} = d\omega$.

$$X = 1 - 2n \tan \omega \cos a + n^2 \tan^2 \omega$$
$$= (\cos^2 \omega - 2n \sin \omega \cos \omega \cos a + n^2 \sin^2 \omega) \div \cos^2 \omega$$
$$= \{(1 + n^2) - 2n \sin 2\omega \cos a + (1 - n^2).\cos 2\omega\} \div 2 \cos^2 \omega$$
$$= (1 - \sin 2\nu \sin 2\omega \cos a + \cos 2\nu.\cos 2\omega) \div 2 \cos^2 \nu . \cos^2 \omega.$$

Let μ, β be taken such that $\left.\begin{array}{l}\sin \mu \sin \beta = \sin 2\nu \cos a \\ \sin \mu \cos \beta = \cos 2\nu\end{array}\right\}$;

$$\therefore \quad \cos \mu = \sin 2\nu \sin a, \text{ and } \tan \beta = \tan 2\nu.\cos a.$$

Also $X = \{1 + \sin \mu (\cos 2\omega \cos \beta - \sin 2\omega \sin \beta)\} \div 2 \cos^2 \nu.\cos^2 \omega$,

$$= (1 + \sin \mu \cos \theta) \div 2 \cos^2 \nu \cos^2 \omega; \text{ if } \theta = 2\omega + \beta:$$

whence $X_1 = \int \log X . d\omega = \frac{1}{2} \int \log (1 + \sin \mu \cos \theta)\, d\theta$

$$- \omega \log (2 \cos^2 \nu) - 2\daleth (\tfrac{1}{2}\pi - \omega). \ldots (5):$$

in which the remaining integral has but one arbitrary constant.

Farther, let $m = \tan \frac{1}{2}\mu$, or $\sin \mu = 2m \div (1 + m^2) = 2m \cos^2 \frac{1}{2}\mu$;

$\therefore \log (1 + \sin \mu \cos \theta) = \log (1 + 2m \cos \theta + m^2) + 2 \log \cos \frac{1}{2}\mu$.

Assume η such that $\tan \eta = \sin \theta \div (m + \cos \theta)$,

$$\text{or } m = \sin (\theta - \eta) \div \sin \eta,$$

$$\therefore \; 1 + 2m \cos \theta + m^2 = \sin^2 \theta + (m + \cos \theta)^2$$
$$= \sin^2 \theta + \left(\frac{\sin \theta}{\tan \eta}\right)^2 = \left(\frac{\sin \theta}{\sin \eta}\right)^2.$$

whence $\frac{1}{2} \int \log (1 + \sin \mu \cos \theta)\, d\theta$

$$= \int \{\log \sin \theta - \log \sin \eta + \log \cos \tfrac{1}{2}\mu\}\, d\theta$$
$$= - \daleth\theta - \int \log \sin \eta . d\theta + \theta . \log . \cos \tfrac{1}{2}\mu.$$

Now $\int \log \sin \eta . d\theta$

$$= \int \log . \frac{\sin (\theta - \eta)}{m} . d\theta = \int \log \frac{\sin (\theta - \eta)}{m} . \{d(\theta - \eta) + d\eta\}$$
$$= - \daleth (\theta - \eta) - (\theta - \eta) \log m + \int \log \sin \eta . d\eta$$
$$= - \daleth (\theta - \eta) - (\theta - \eta) \log \tan \tfrac{1}{2}\mu - \daleth\eta.$$

Therefore
$$\left.\begin{array}{l}\frac{1}{2} \int \log (1 + \sin \mu \cos \theta)\, d\theta \\ = \daleth (\theta - \eta) + \daleth\eta - \daleth\theta + \theta \log \sin \frac{1}{2}\mu - \eta \log \tan \frac{1}{2}\mu\end{array}\right\} \ldots (6),$$

which is a general formula, provided that $\tan \eta = \dfrac{\sin \theta}{\tan \frac{1}{2}\mu + \cos \theta}$;

and completes the reduction of X_1 to the function $\daleth$.

[redacted]

3. The integral X_2 remains. Using for X the same transformation as before, let us write 2θ in place of θ, so that now $2\theta = 2\omega + \beta$. We have, moreover,

$$\frac{xdx}{x^2 + n^2} = \tfrac{1}{2} d \log (n^2 + x^2) = \tfrac{1}{2} d \log \sec^2 \omega = \tan \omega d\omega$$

$$\text{and also} = - \tfrac{1}{2} d \log (2 \cos^2 \nu \,.\, \cos^2 \omega).$$

Whence

$$X_2 = \int \{\log (1 + \sin \mu \cos 2\theta) - \log (2 \cos^2 \nu \,.\, \cos^2 \omega)\} \frac{xdx}{x^2 + n^2}$$

$$= \int \log (1 + \sin \mu \cos 2\theta) \tan \omega d\omega + \tfrac{1}{4} \log^2 . (2 \cos^2 \nu \,.\, \cos^2 \nu).$$

Put $b = \tan \frac{1}{2}\beta$; $t = \tan \theta$, $d\omega = d\theta = \dfrac{dt}{1 + t^2}$:

$$\cos 2\theta = \frac{1 - t^2}{1 + t^2}; \ \tan \omega = \tan \left(\theta - \frac{b}{2}\right) = \frac{t - b}{1 + bt};$$

$$\text{and } \tan \omega . d\omega = \frac{t - b}{1 + bt} \cdot \frac{dt}{1 + t^2} = \frac{tdt}{1 + t^2} - \frac{bdt}{1 + bt}.$$

Hence the integral which remains, becomes

$$\left\{\int \log \{1 + t^2 + \sin \mu . (1 - t^2)\} - \log (1 + t^2)\right\} . \left(\frac{tdt}{1 + t^2} - \frac{bdt}{1 + bt}\right).$$

Write

$$T_1 = \int \log \{1 + \sin \mu + (1 - \sin \mu) . t^2\} . \tfrac{1}{2} d \log (1 + t^2),$$

$$T_2 = \int \log (1 + t^2) . \tfrac{1}{2} d \log (1 + t^2) = \tfrac{1}{4} \log^2 (1 + t^2) = \log^2 \cos \theta,$$

$$T_3 = \int \log \{1 + \sin \mu + (1 - \sin \mu) . t^2\} \, d \log (1 + bt),$$

$$T_4 = \int \log (1 + t^2) \, d \log (1 + bt).$$

Then $X_2 = T_1 - T_2 - T_3 + T_4 + \frac{1}{4} \log^2 . (2 \cos^2 \nu \,.\, \cos^2 \omega)$.

To find T_1, let $1 + t^2 = mv$, and $m = 2 \sin \mu \div (1 - \sin \mu)$;

$$\therefore \quad T_1 = \tfrac{1}{2} \int l \{2 \sin \mu . (1 + v) \, dl (mv)$$

$$= \tfrac{1}{2} l (2 \sin \mu) \, l (mv) + \tfrac{1}{2} L (1 + v);$$

where $mv = \sec^2 \theta$,

$$1 + v = \frac{1 + t^2 + \sin \mu (1 - t^2)}{2 \sin \mu} = \frac{\sec^2 \theta (1 + \sin \mu \cos 2\theta)}{2 \sin \mu};$$

so that $T_1 = - \log (2 \sin \mu) \log \cos \theta + \frac{1}{2} L \dfrac{1 + \sin \mu \cos 2\theta}{\sin \mu (1 + \cos 2\theta)}$.

For T_3, put $1 + bt = kz$, $1 + \sin \mu + (1 - \sin \mu) t^2$

$= b^{-2} . \{1 + b^2 - (1 - b^2) \sin \mu - 2kz (1 - \sin \mu) + k^2 z^2 (1 - \sin \mu)\}$.

Take k such that $k^2 (1 - \sin \mu) = 1 + b^2 - (1 - b^2) \sin \mu$;

or $= (1 + b^2) \{1 - \cos \beta \sin \mu\} = \sec^2 \frac{1}{2}\beta \, (1 - \cos 2\nu)$;

$$\therefore \quad k = \sec\frac{\beta}{2}\sqrt{\frac{1-\cos 2\nu}{1-\sin\mu}}\,.$$

$$\text{Also let } \cos\gamma = k^{-1} = \cos\frac{\beta}{2}\sqrt{\frac{1-\sin\mu}{1-\cos 2\nu}}\,,$$

and observe that $b^{-2}k^2(1-\sin\mu) = (\sin\tfrac{1}{2}\beta)^{-2}.\,(1-\cos 2\nu)$,

also $kz = 1 + bt = 1 + \tan\tfrac{1}{2}\beta\tan\theta = \dfrac{\cos(\theta-\frac{1}{2}\beta)}{\cos\frac{1}{2}\beta\cos\theta} \propto \dfrac{\cos\omega}{\cos\theta}$.

Hence

$$T_3 = \int\log\left\{(\sin\tfrac{1}{2}\beta)^{-2}.(1-\cos 2\nu).(1-2z\cos\gamma+z^2)\right\}d\log(kz)$$

$$= \log\left\{(\sin\tfrac{1}{2}\beta)^{-2}.\,(1-\cos 2\nu)\right\}\log\frac{\cos\omega}{\cos\theta} + 2\lambda\,(z,\gamma).$$

From this we may deduce T_4 by momentarily supposing $\mu = 0$, which makes $\cos 2\nu = 0$; so that, writing $k'y$ for kz, we get $k' = \sec\tfrac{1}{2}\beta$, and γ changes into $\tfrac{1}{2}\beta$. Also $y = \dfrac{\cos\omega}{\cos\theta}$.

$$\therefore \quad T_4 = -\log\sin^2\tfrac{1}{2}\beta\log\frac{\cos\omega}{\cos\theta} + 2\lambda\,(y,\tfrac{1}{2}\beta),$$

and $-T_3 + T_4$

$$= -\log(1-\cos 2\nu)\log\frac{\cos\omega}{\cos\theta} + 2\lambda\,(y,\tfrac{1}{2}\beta) - 2\lambda\,(z,\gamma);$$

in which we may deduce z, γ from $y, \tfrac{1}{2}\beta$ by writing

$$c^2 = \frac{1-\sin\mu}{1-\cos 2\nu}, \quad z = cy, \quad \cos\gamma = c.\cos\tfrac{1}{2}\beta.$$

Combining all the results, we have to observe that (neglecting constants)

$$\tfrac{1}{4}l^2(2\cos^2\nu\cos^2\omega) - l(2\sin\mu)\,l\cos\theta - l^2\cos\theta$$
$$- l(1-\cos 2\nu)(l\cos\omega - l\cos\theta)$$
$$= \log^2(n\sec\omega) - \log^2\left(\cos\theta.\frac{\sqrt{\sin\mu}}{\sin\nu}\right).$$

Whence, finally,

$$\left.\begin{aligned} X_2 &= \log^2(n\sec\omega) - \log^2\left(\cos\theta.\frac{\sqrt{\sin\mu}}{\sin\nu}\right) \\ &+ \tfrac{1}{2}L\,\frac{1+\sin\mu\cos 2\theta}{\sin\mu\,(1+\cos 2\theta)} + 2\lambda\left(y,\frac{\beta}{2}\right) - 2\lambda\,(z,\gamma) \end{aligned}\right\} \ldots(7).$$

Observe that $ny^{-1} = n\cos\tfrac{1}{2}\beta - x\sin\tfrac{1}{2}\beta$; and the quantity under L may also be denoted by $\dfrac{y^2X\cos^2\nu}{\sin\mu}$. The result thus obtained admits likewise of other forms, by means of the

properties of λ and Λ; but all that is here aimed at, is to shew the possibility of the reduction.

It is easy to verify our result, in the case of $a = \frac{1}{2}\pi$. On the whole it has appeared that the integral $\int F_1 x \log F_2 x dx$ contains only three elementary forms, which we have denoted by L, $\mathfrak{h}$, Λ. It is proposed to call these Logarithmic Integrals *of the Second Order.*

4. Before leaving the integrals X_1, X_2, it may be well to examine the special cases of $n = 1$, and of $x = \infty$. First, to find X_1 when $x = \infty$.

Put $X' = \int_0 \tan^{-1}\frac{x}{n}\,.\,d\log X = \log X\,.\tan^{-1}\frac{x}{n} - X_1$:

$$\therefore\ \frac{dX'}{dn} = -\log X\,.\,\frac{x}{n^2+x^2} - \frac{dX_1}{dn};$$

$$\text{and when } x = \infty,\ \frac{dX'}{dn} = -\frac{dX_1}{dn}.$$

Again, $\frac{dX'}{dn} = \int_0 \frac{-x}{n^2+x^2}\,.\,d\log X$; which we assume

$$= \int_0 \left\{\frac{2px+2q}{x^2+n^2} + \frac{2r(x-\cos a)+2s\sin a}{x^2-2x\cos a+1}\right\}dx;$$

and by common methods we find that if $N = n^4 + 2n^2\cos 2a + 1$,

$$p = -r = \frac{\cos a.(n^2+1)}{N};\ q = -\frac{n(n^2+\cos 2a)}{N};\ s = \frac{(n^2-1)\sin a}{N}.$$

Also $\frac{dX'}{dn} = -p\log(n^2) + p\log\frac{x^2+n^2}{X}$

$$+ 2q\tan^{-1}\frac{x}{n} + 2s\,.\tan^{-1}.\,\frac{x\sin a}{1-x\cos a}.$$

Let $x = \infty$; then integrating for n, observing that $\frac{dX'}{dn} = -\frac{dX_1}{dn}$,

$$-X_1 = -\int_0 2\log n\,.pdn + \pi\int_0 qdn + 2(\pi - a)\int_0 sdn;$$

observing that as X_1 vanishes with n, no function of x is to be added. Now

$$2pdn = \frac{2\cos a\,(n^2+1)\,dn}{n^4+2n^2\cos 2a+1} = \frac{\cos a\,.dn}{n^2-2n\sin a+1} + \frac{\cos a\,dn}{n^2+2n\sin a+1};$$

$$\therefore \text{ if } \tan\rho = \frac{n\cos a}{1-n\sin a},\ \text{and } \tan\sigma = \frac{n\cos a}{1+n\sin a}$$

$$\int_0 2\log n.pdn = \int_0\frac{\cos a\,.\log n\,.\,dn}{n^2-2n\sin a+1} + \int_0\frac{\cos a\,.\log n\,.\,dn}{n^2+2n\sin a+1}$$

$$= [\mathfrak{h}\{\rho+(\tfrac{1}{2}\pi-a)\} - \mathfrak{h}\rho - \mathfrak{h}(\tfrac{1}{2}\pi-a)] + [\mathfrak{h}\{\sigma+(\tfrac{1}{2}\pi+a)\} - \mathfrak{h}\sigma - \mathfrak{h}(\tfrac{1}{2}\pi+a)].$$

which is a simpler expression than would arise from putting $n = 1$ in equation (7).

§ II.—*On Spence's Integral* $\displaystyle\int_1 \frac{\log x\,dx}{x-1}$.

5. Spence has tabulated this integral, on the assumption that x is positive; and this suffices in practice. Yet it embarrasses us in generalizing concerning the integrals which are partially reducible to L, not to be at liberty to suppose x negative. Supposing $\log x$ to have arisen out of integration, and to be $= \displaystyle\int\frac{dx}{x}$, no imaginary quantity results from regarding x as negative: in fact, we may look on *log* x as a short mode of writing $\frac{1}{2}\log x^2$; then, in passing through 0, x produces no discontinuity in L.

The following are the chief properties of L, which are easily verified:

$$Lx + L(-x) = \tfrac{1}{2}L(x^2) + \tfrac{3}{2}L0,$$

$$L(\pm x) + L(1 \mp x) = \log x \,.\log(1 \mp x) + L0,$$

$$Lx + Lx^{-1} = \tfrac{1}{2}\log^2 x \quad (x \text{ positive}),$$

$$L(1+x) + L(1-x) = \tfrac{1}{2}L(1-x^2),$$

$$L(1+x) + L(1+x^{-1}) = \tfrac{1}{2}\log^2 x + C;$$

where $C = 2L2$, if x is positive; but $C = 2L0$, if x is negative. This is proved by making $x = 1$ in the former case, and $x = -1$ in the latter. The discontinuity is occasioned by $L(1+x^{-1})$ becoming infinite, when x is passing through 0. So, if we wish to make x negative in the third formula, we must add $2L(-1)$ or $-\frac{1}{2}\pi^2$ on the right-hand side. Farther, we have

$$-L0 = 2L2 = \tfrac{1}{6}\pi^2, \quad L(-1) = -3L2 = -\tfrac{1}{4}\pi^2.$$

When $(x-1)$ is infinitesimal,

$$Lx = x - 1, \quad \text{and } \tfrac{1}{4}\pi^2 + L(-x) = \left(\frac{x-1}{2}\right)^2.$$

When x is large,

$$L(-x+1) = 2L0 + \tfrac{1}{2}\log^2 x + 1^{-2}x^{-1} + 2^{-2}x^{-2} + 3^{-2}x^{-3} + 4^{-2}x^{-4} + \&c. \ldots$$

If we desire to know $L(-x)$ numerically, we may either calculate it by the last formula, or (when x is not large) deduce it by the first or second of the equations from Spence's Table.

In future I shall always employ $\log x$ as a mere representation of $\displaystyle\int\frac{dx}{x}$ or $\frac{1}{2}\log(x^2)$; and it will only be necessary,

in correcting integrals, to observe whether the arbitrary constant is altered by supposing the quantity under *log* to pass from positive to negative.

§. III.—*On the integral*, $-\int_0 \log \sin x\, dx$.

6. Since $\log \sin x$ and $\log \sin(-x)$ are by hypothesis the same, or to speak otherwise, since $\natural(x) = -\tfrac{1}{2}\int_0 \log \sin^2 x\, dx$,

$$\therefore\quad \natural(-x) = -\natural x. \ldots\ldots\ldots\ldots (11).$$

Also $\natural(n\pi \pm x) = \mp \int \log \sin(n\pi \pm x)\, dx = \mp \int \log \sin x\, dx$,

$$\text{or}\quad \natural(n\pi \pm x) = \natural(n\pi) \pm \natural x.$$

Make n successively $1, 2, 3, \ldots$ and $x = \pi$ } and we find $\natural(n\pi) = n\natural\pi$.

Hence it readily follows that

$$\left.\begin{array}{r} \natural(n\pi \pm x) = n\natural\pi \pm \natural x \\ \natural(\pi - x) = \natural\pi - \natural x;\quad 2\natural\,\tfrac{1}{2}\pi = \natural\pi \end{array}\right\} \ldots\ldots (12).$$

These equations indicate, that to tabulate $\natural$ from $x = 0$ to $x = \frac{1}{2}\pi$ will suffice.

7. To find $\natural\pi$.

Since $-\log(2\sin x)$

$$= \cos 2x + 2^{-1}\cos 4x + 3^{-1}\cos 6x + 4^{-1}\cos 8x + \&c.,$$

therefore $\natural x = x\log 2$

$$+ \tfrac{1}{2}\left\{1^{-2}\sin 2x + 2^{-2}\sin 4x + 3^{-2}\sin 6x + \&c.\ldots\right\}\ldots. (13).$$

Hence $\quad \natural\pi = \pi\log 2 = 2{\cdot}177586\ 0933046.$

Also $\quad \natural\,\tfrac{1}{4}\pi = \tfrac{1}{4}\natural\pi + \tfrac{1}{2}\left\{1^{-2} - 3^{-2} + 5^{-2} - 7^{-2} + 9^{-2} - \&c.\right\}.$

8. Since $\sin 2x = 2\sin x \sin(\frac{1}{2}\pi - x)$, take logs. and integrate;

$$\therefore\quad \tfrac{1}{2}\natural(2x) = (\tfrac{1}{2}\pi - x)\log 2 + \natural x - \natural(\tfrac{1}{2}\pi - x) \ldots (14).$$

We may generalize this theorem. Since

$$\sin nx = 2^{n-1}.\sin x.\sin\left(\frac{\pi}{n} + x\right).\sin\left(\frac{2\pi}{n} + x\right)\ldots\sin\left(\frac{n-1}{n}\pi + x\right),$$

take the logarithms, as before, and integrate;

$$\therefore\quad \frac{1}{n}\natural(nx) = C - (n-1)x\log 2 + \natural x + \natural\left(\frac{\pi}{n} + x\right) + \ldots.$$

$$+ \natural\left(\frac{n-1}{n}\pi + x\right):$$

To find C, make $x = 0$;

$$\therefore\quad -C = \natural\frac{\pi}{n} + \natural\frac{2\pi}{n} + \ldots. + \natural\frac{n-1}{n}\pi.$$

1. To expand $\text{ϟ}x$ in converging series, when x does not [exce]ed 30°.

[Fi]rst, put $\sin x = y$,

$$\therefore \quad \text{ϟ}x = -x\log y + \int \sin^{-1} y \,.\, y^{-1} dy.$$

[Expa]nd $\sin^{-1} y$ and integrate. There results

$$[\text{ϟ}x] = -x\log x + 1^{-2}\sin x + \tfrac{1}{2}\,.\,3^{-2}\sin^3 x + \frac{1.3}{2.4}\,.\,5^{-2}\sin^5 x + \&c. \;\ldots(18).$$

Thus, in particular, if $x = \frac{1}{6}\pi$,

$$\text{ϟ}\,30^\circ = \tfrac{1}{6}\text{ϟ}\pi + 1^{-2}.\,2^{-1} + \tfrac{1}{2}\,.\,3^{-2}.\,2^{-3} + \frac{1.3}{2.4}\,5^{-2}.\,2^{-5} + \&c. \ldots$$

Next, let $S_n = 1^{-n} + 2^{-n} + 3^{-n} + \&c.$, a known sum; and $x = \pi\omega$;

$$\therefore \quad \log\sin(\pi\omega) = \log(\pi\omega) - S_2\frac{\omega^2}{1} - S_4\frac{\omega^4}{2} - S_6\frac{\omega^6}{3} - \&c. \ldots$$

Integrate:

$$\frac{1}{\pi}\text{ϟ}(\pi\omega) = \omega\{1 - \log \pi\omega\} + S_2\,.\,\frac{\omega^3}{1.3} + S_4\,.\,\frac{\omega^5}{2.5} + S_6\,.\,\frac{\omega^7}{3.7} + \&c. \;\ldots\ldots(19).$$

To increase the convergence, add to the penultimate series before integration:

$$-\log(1-\omega^2) = \omega^2 + \tfrac{1}{2}\omega^4 + \tfrac{1}{3}\omega^6 + \ldots \&c.$$

$$\therefore \quad -\log\sin(\pi\omega) + \log(1-\omega^2)$$

$$= -\log(\pi\omega) + (S_2 - 1)\frac{\omega^2}{1} + (S_4 - 1)\tfrac{1}{2}\omega^4 + (S_6 - 1)\tfrac{1}{3}\omega^6 + \&c.$$

whence

$$\left.\begin{aligned}\frac{1}{\pi}\text{ϟ}(\pi\omega) &= \omega\{3 - \log\pi\omega - \log(1-\omega) - \log(1+\omega)\} - \log\frac{1+\omega}{1-\omega} \\ &\quad + (S_2 - 1)\frac{\omega^3}{1.3} + (S_4 - 1)\frac{\omega^5}{2.5} + (S_6 - 1)\frac{\omega^7}{3.7} + \&c. \ldots\end{aligned}\right\} \ldots(20),$$

$$\int_0 \tan x \,.\, x dx = \int_0 \frac{x \tan^{-1} x \,.\, dx}{1 + x^2} = A_1 \tfrac{1}{3} x^3 - A_2 \,.\, \tfrac{1}{5} x^5 + A_3 \tfrac{1}{7} x^7 - \&c. \ldots$$

$$= \{\tfrac{1}{3} x^3 - \tfrac{1}{5} x^5 + \tfrac{1}{7} x^7 - \&c. \ldots\} - \tfrac{1}{3} \{\tfrac{1}{5} x^5 - \tfrac{1}{7} x^7 + \tfrac{1}{9} x^9 - \&c.\}$$
$$+ \tfrac{1}{5} \{\tfrac{1}{7} x^7 - \tfrac{1}{9} x^9 + \&c. \ldots\} - \tfrac{1}{7} \&c. \&c. \ldots \ldots$$

us henceforth use $\phi_n x$ for $\int_0 \tan^{n-1} x \,.\, dx$,

$$\text{or} \quad \phi_n x = \frac{\tan^n x}{n} - \frac{\tan^{n+2} x}{n+2} + \frac{\tan^{n+4} x}{n+4} - \&c. \ldots \ldots$$

hat $\phi_1 x = x$; $\phi_2 x = \log \sec x$; and $\phi_{n+2} x = \dfrac{\tan^n x}{n} - \phi_n x$:

and we finally obtain

$$\text{ϟ}(\tfrac{1}{2}\pi - x) = \text{ϟ}\,\tfrac{1}{2}\pi + x \log \cos x$$
$$+ \phi_3 x - \tfrac{1}{3} \phi_5 x + \tfrac{1}{5} \phi_7 x - \tfrac{1}{7} \phi_9 x + \&c. \ldots \ldots (24),$$

When x is $< 10°$, $\phi_7 x$ will not affect the sixth decimal.

To obtain a more converging series, let $v = 1 - \cos x$.

$$\int_0 \tan x \,.\, x dx = \int - x \,.\, d \log \cos x = \int_0 \frac{\cos^{-1}(1 - v) \,.\, dv}{1 - v}.$$

Now $\cos^{-1}(1 - v) = \sqrt{(2v)} \left\{1 + \dfrac{1}{2.3} \,.\, (\tfrac{1}{2} v) + \dfrac{1.3}{2.4.5} \,.\, (\tfrac{1}{2} v)^2 + \&c.\right\}$.

Let, then, $\dfrac{\cos^{-1}(1 - v)}{1 - v} = \sqrt{(2v)} \{B_1 + B_2 v + B_3 v^2 + \&c. \ldots\}$

and we get $B_1 = 1$, $B_2 = B_1 + \tfrac{1}{2} . \tfrac{1}{3} 2^{-1}$; $B_3 = B_2 + \dfrac{1.3}{2.4} . \tfrac{1}{5} 2^{-2}$; &c. . .

whence $B_\infty = 1 + \tfrac{1}{2} . \tfrac{1}{3} 2^{-1} + \dfrac{1.3}{2.4} . \tfrac{1}{5} 2^{-2} + \dfrac{1.3.5}{2.4.6} . \tfrac{1}{7} 2^{-3} + \&c. \ldots$

$$= \frac{1}{\sqrt{2}} \cos^{-1} . \{1 - 1\} = \frac{\pi}{2\sqrt{2}}.$$

To increase therefore the convergence, put $C_n = \dfrac{\pi}{2\sqrt{2}} - B_n$;

so that $\qquad C_{n+1} - C_n = B_n - B_{n+1}$;

$\Delta \int Fx\, dx = h \{Fx + m_1 \Delta Fx + m_2 \Delta^2 Fx + \text{\&c.} \ldots\}$

by slightly modifying the process, it is easy to shew that have also

$$;dx = h\,\{F(x + h) - m_1 \Delta Fx$$
$$+ m_2 \Delta^2 F(x - h) - m_3 \Delta^3 F(x - 2h) + \text{\&c.}\},$$

re F is any function whatever. Here we assume

$$Fx = -\log \sin x:$$
$$m_1 = \tfrac{1}{2},\quad m_2 = -\tfrac{1}{12},\quad m_3 = \tfrac{1}{24};$$

$$= -h\,\{l \sin x + \tfrac{1}{2}\Delta l \sin x - \tfrac{1}{12}\Delta^2 l \sin x + \tfrac{1}{24}\Delta^3 l \sin x\} \ldots (27),$$

$$x = -h\,\{l \sin (x + h) - \tfrac{1}{2}\Delta l \sin x$$
$$- \tfrac{1}{12}\Delta^2 l \sin (x - h) - \tfrac{1}{24}\Delta^3 l \sin (x - 2h)\} \quad .(28).$$

e the sum:

$$\therefore\ 2\Delta \daleth x = -h\,[l \sin (x + h) - \tfrac{1}{12}\{\Delta^2 l \sin x + \Delta^2 l \,.\, \sin (x - h)\}$$
$$+ l \sin x + \tfrac{1}{24}\{\Delta^3 l \sin x - \Delta^3 l \sin (x - 2h)\}];$$

or, when the last term is negligible,

$$\Delta \daleth x = -\tfrac{1}{2}h\,[l \sin (x + h) + l \sin x - \tfrac{1}{12}\{\Delta^2 l \sin x$$
$$+ \Delta^2 l \sin .\, (x - h)\}] \ldots (29).$$

But perhaps certain series of Legendre's are better still.

Let $M_1, M_2, M_3, \ldots$ be determined by the equation

$$\tfrac{1}{2}x \div \sin^{-1}(\tfrac{1}{2}x) = 1 + M_1 x^2 + M_2 x^4 + M_3 x^6 + \ldots$$

then $\quad M_1 = \dfrac{1}{24},\quad M_2 = -\dfrac{17}{24^2 .\, 10},\quad M_3 = \dfrac{367}{8^2 .5.6.7.8.9};$

and we have

$$\Delta \int Fx\, dx = h\,\{F(x + \tfrac{1}{2}h) + M_1 \Delta^2 F(x - \tfrac{1}{2}h) + M_2 \Delta^4 F(x - \tfrac{1}{2}3h) + \text{\&c.}\ldots\}$$

also $\quad = h\,\{F(x + \tfrac{1}{2}h) + M_1 \Delta^2 F(x + \tfrac{1}{2}3h) + M_2 \Delta^4 F(x + \tfrac{1}{2}5h) + \text{\&c.}\ldots\}$

which here give

$$\left.\begin{aligned}
&\Delta \daleth x = -h\,\{l \sin (x + \tfrac{1}{2}h) \\
&\quad + \tfrac{1}{24}\Delta^2 l \sin (x - \tfrac{1}{2}h) - \tfrac{17}{5760}\Delta^4 l \sin (x - \tfrac{1}{2}3h) + \text{\&c.}\} \\
&\text{also } = -h\,\{l \sin (x + \tfrac{1}{2}h) \\
&\quad + \tfrac{1}{24}\Delta^2 l \sin (x + \tfrac{1}{2}3h) - \tfrac{17}{5760}\Delta^4 l \sin (x + \tfrac{1}{2}5h) + \text{\&c.}\}
\end{aligned}\right\} \ldots (30),$$

which are easy to us, because we have tables of *log sin.*

last falls under Ex. (7), as a particular case, when $\frac{1}{2}\pi$.

$- 2r \cos \alpha \sin \omega + r^2 \sin^2 \omega)\, d\omega$, this also by the following process.

$x = \tan \frac{1}{2}\omega$, or $\sin \omega = \dfrac{2x}{1 + x^2}$: and

$$\therefore \Omega = \int_0 \log \{1 - 4rx \cos \alpha + (4r^2 + 2)\, x^2 - 4rx^3 \cos \alpha + x^4\}\, d\omega$$
$$+ 8\natural\left(\frac{\pi - \omega}{2}\right).$$

Assume the quantity under *log* to be

$$= (1 - 2nx \cos \gamma + n^2x^2)\,(1 - 2n^{-1}\, x \cos \gamma + n^{-2}x^2);$$

$\therefore (n + n^{-1}) \cos \gamma = 2r \cos \alpha$, and $4r^2 + 2 = 4 \cos^2 \gamma + n^2 + n^{-2}$.

Let $\tan \nu = n$, $\tan \rho = r$; $\therefore 4r^2 + 4 = 4 \cos^2 \gamma + (n + n^{-1})^2$;

or $\cos \gamma = r \cos \alpha \sin 2\nu$, and $2r \operatorname{cosec} 2\rho = \cos^2 \gamma + \operatorname{cosec}^2 2\nu$.

Eliminate γ, and solve for $\operatorname{cosec} 2\nu$. The result is, that if we take $\sin 2\zeta = \cos \alpha \sin 2\rho$, and select that root of ζ which makes $\pm \sin \zeta$ least, we have

$$\sin 2\nu = \frac{\cos \rho}{\cos \zeta}, \text{ and } \cos \gamma = \frac{\sin \zeta}{\cos \rho}.$$

Hence, having found ν and γ,

$$\left.\begin{aligned} \text{let } \Omega' &= \int_0 \log (1 - 2n \cos \gamma \tan \tfrac{1}{2}\omega + n^2 \tan^2 \tfrac{1}{2}\omega)\, \tfrac{1}{2} d\omega \\ \Omega'' &= \int_0 \log (1 - 2n^{-1} \cos \gamma \tan \tfrac{1}{2}\omega + n^{-2} \tan^2 \tfrac{1}{2}\omega)\, \tfrac{1}{2} d\omega \end{aligned}\right\}$$

where we pass from Ω' to Ω'' by changing ν to $(\frac{1}{2}\pi - \nu)$; then Ω becomes $= 2\Omega' + 2\Omega'' + 8\natural\left(\dfrac{\pi - \omega}{2}\right)$.

$$\therefore\quad V_{0,0} = \int \log(1+y^2)\,\frac{dy}{y} - \int \log(2y)\,\frac{dy}{y}$$

$$= \tfrac{1}{2} L(1+y^2) - \tfrac{1}{2}\log^2(2y) + \text{const.}$$

:n $a = 1$ and $b = 1$, let $y = x + \sqrt{(x^2+1)}$:

$$V_{0,0} = \log y \,.\, \log(y^2 - 1) - \tfrac{1}{2} L(y^2) - \tfrac{1}{2}\log^2(2y) + \text{const.}$$

recapitulate then: $V_{m,n}$ is always reducible to ל or L; in particular, $\int \frac{x^{2m-1}\log x\,dx}{(a+bx^2)^{n+\frac{1}{2}}}$ can be found by circular and logarithms;

$\int \frac{x^{2m}\log x\,dx}{(1-x^2)^{n+\frac{1}{2}}}$ is reducible to ל; $\int \frac{x^{2m}\log x\,dx}{(x^2\pm 1)^{n+\frac{1}{2}}}$ to L;

$\int \frac{x^{-2m+1}\log x\,dx}{(x^2-1)^{n+\frac{1}{2}}}$ to ל; $\int \frac{x^{-2m+1}\log x\,dx}{(1\pm x^2)^{n+\frac{1}{2}}}$ to L;

where m and n are integers, m positive and n either positive or negative.

§. V.—*On the Higher Transcendents derivable from* ל.

16. Spence has imagined the integrals L^3, L^4, L^5. . deduced from L^2 or L, by the law $L^n(1+x) = \int L^{n-1}(1+x)\,x^{-1}\,dx$; and has exhibited various fundamental properties of L^n. Put

$$x = e^{2\omega}, \quad \therefore\ e^{2\omega} + 1 = (e^{\omega} + e^{-\omega})\,e^{\omega},$$

$$\text{or}\ \log(x+1) = \log(\epsilon^{\omega} + \epsilon^{-\omega}) + \omega, \quad \text{and}\ x^{-1}dx = 2d\omega;$$

$$\therefore\ L(1+x) = \int \log(\epsilon^{\omega} + \epsilon^{-\omega})\,2d\omega + \omega^2.$$

When ω changes to $\omega\sqrt{(-1)}$, the integral here becomes $2\sqrt{(-1)}\,.\int\log(2\cos\omega)\,d\omega$, which exhibits the relation which exists between L and ל by imaginaries.

17. Since $d\omega \propto x^{-1}dx$, we may imagine a series of functions $\text{ל}^3, \text{ל}^4, \text{ל}^5$. . analogous to L^3, L^4, L^5. ., by the law $\text{ל}^3 x = \int_0 \text{ל}\, x\,dx$, and generally $\text{ל}^n x = \int_0 \text{ל}^{n-1} x\,dx$; and we now regard ל as virtually ל^2. Write also

$$\lambda_1 x = -\int_0 \log x\,dx, \quad \lambda_n x = \int_0 \lambda_{n-1}\,x\,dx;$$

then $\lambda_2 x = x(1 - \log x)$,

$$\lambda_{n+1} x = \frac{x^n}{1.2\ldots n}\left\{1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots\ldots + \frac{1}{n} - \log x\right\} \ldots (33).$$

Also, as $\text{ϟ}^2 x = \lambda_2 x + H_1 \dfrac{x^3}{1.3} + H_2 \dfrac{x^5}{2.5} + H_3 . \dfrac{x^7}{3.7} + \&c.\ldots\ldots$

if H_n stands for $\pi^{-2n} S_{2n}$; (see equation 19.)

$$\therefore \text{ϟ}^n x = \lambda_n x + \frac{2 H_1 x^{n+1}}{2.3\ldots(n+1)} + \frac{2 H_2 x^{n+3}}{4.5\ldots(n+3)} + \frac{2 H_3 x^{n+4}}{6.7\ldots(n+5)} + \&c. \ldots\ldots(34).$$

18. Since $2\text{ϟ}^2(\tfrac{1}{2}x) = xl2 + 1^{-2}\sin x + 2^{-2}\sin 2x + 3^{-2}\sin 3x + \&c.\ldots$ perpetual integration, with suitable addition of constants, gives

$$\left.\begin{aligned} 2^{2n-1}.\text{ϟ}^{2n}(\tfrac{1}{2}x) &= \frac{x^{2n-1}l2}{1.2\ldots(2n-1)} \\ &+ \frac{x^{2n+3}S_3}{1.2\ldots(2n-3)} - \frac{x^{2n-5}S_5}{1.2\ldots(2n-5)} + \&c.\ldots \pm \frac{x}{1} S_{2n-1} \\ &- 1^{-2n}\sin(x - 2n.\tfrac{1}{2})\pi - 2^{-2n}\sin(2x - 2n\tfrac{1}{2}\pi) \\ &\qquad - 3^{-2n}\sin(3x - 2n\tfrac{1}{2}\pi) - \&c. \end{aligned}\right\} \ldots (35).$$

$$\left.\begin{aligned} 2^{2n}\text{ϟ}^{2n+1}(\tfrac{1}{2}x) &= \frac{x^{2n}l2}{1.2..2n} \\ &+ \frac{x^{2n-2}S_3}{1.2..(2n-2)} - \frac{x^{2n-4}S_5}{1.2..(2n-4)} + \&c.\ldots \mp S_{2n+1} \\ &- 1^{-2n-1}\sin\left\{x - (2n+1).\tfrac{1}{2}\pi\right\} \\ &\qquad - 2^{-2n-1}\sin\left\{2x - (2n+1)\tfrac{1}{2}\pi\right\} - \&c.\ldots \end{aligned}\right\} \ldots (36).$$

And if in these we put $x = 2\pi$, we get

$$\text{ϟ}^3\pi = \frac{\pi^2 l2}{1.2}:\quad \text{ϟ}^4\pi = \frac{\pi^3 l2}{1.2.3} + 2^{-2}.\frac{\pi}{1} S_3:$$

$$\text{ϟ}^5\pi = \frac{\pi^4 l2}{1.2.3.4} + 2^{-2}.\frac{\pi^2}{1.2}.S_3:$$

$$\text{ϟ}^6\pi = \frac{\pi^5 l2}{1.2..5} + 2^{-2}.\frac{\pi^3}{1.2.3} S_3 - 2^{-4}.\frac{\pi}{1} S_5:$$

$$\text{ϟ}^7\pi = \frac{\pi^6 l2}{1.2..6} + 2^{-2}.\frac{\pi^4}{1.2.3.4} S_3 - 2^{-4}.\frac{\pi^2}{1.2}.S_5:$$

The law is evident. After the two first terms, the signs are alternate. Thus $\text{ϟ}^n\pi$ is known.

Let $u + u' = a$, it being understood that u and u' are taken ositive. Then, if we assume

$$= \frac{1}{\{\Sigma(\xi - x)^2 + v^2\}^{\frac{1}{2}(s-1)}} - \frac{1}{\{\Sigma(\xi - x)^2 + (2u - v)^2\}^{\frac{1}{2}(s-1)}} \quad \ldots (1),$$

$$= \frac{1}{\{\Sigma(\xi - x')^2 + (a - v)^2\}^{\frac{1}{2}(s-1)}} \quad \ldots\ldots\ldots\ldots (2),$$

have $-2(s-1)\,uU = \left[\int_{-\infty}^{\infty}\right]^s R' \dfrac{dR}{dv}\,[d\xi]^s$, when $v = u$.

is easily seen that the second member of this equation vanishes when $v = \pm\infty$, and that it does not become infinite, even when one of the values 0, $2u$, or a is assigned to u. Hence the preceding equation may be written

$$-2(n-1)\,uU = \int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^s \left(\frac{dR'}{dv}\,\frac{dR}{dv} + R'\,\frac{d^2R}{dv^2}\right)[d\xi]^s\,dv.$$

But we have

$$\int\left[\int_{-\infty}^{\infty}\right]^s \frac{dR'}{dv}\,\frac{dR}{dv}\,[d\xi]^s\,dv = \left[\int_{-\infty}^{\infty}\right]^s \int \frac{dR'}{dv}\,\frac{dR}{dv}\,dv\,[d\xi]^s$$

$$= \left[\int_{-\infty}^{\infty}\right]^s R\,\frac{dR'}{dv}\,[d\xi]^s - \int\left[\int_{-\infty}^{\infty}\right]^s R\,\frac{d^2R'}{dv}\,[d\xi]^s\,dv.$$

When we take the integral with respect to v between the limits $-\infty$ and u, the first term vanishes, since at each limit $R = 0$. Thus the preceding equation is reduced to

$$-2(s-1)\,uU = \int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^s \left(R'\,\frac{d^2R}{dv^2} - R\,\frac{d^2R'}{dv^2}\right)[d\xi]^s\,dv.$$

Now we have $\qquad \dfrac{d^2R'}{dv^2} + \Sigma\,\dfrac{d^2R'}{d\xi^2} = 0,$

for all values of $\xi_1, \xi_2, \ldots$, provided v be not equal to a. Hence this equation is satisfied for all the values of the

$$2(s-1)\,u\,U = P\left(\iint . . \frac{dQ}{dv}\,d\xi_1\,d\xi_2 . . + \iint \frac{dQ}{d\xi_1}\,dv\,d\xi_2 + \&c.\right) . . (4).$$

Let us now assume

$$\xi_1 = v_1 + x_1,\ \xi_2 = v_2 + x_2,\ \&c.,$$

nd

$$v^2 + v_1^2 + . . + v_s^2 = r^2,$$

om which we have

$$Q = \frac{1}{r^{s-1}},\quad \frac{dQ}{dv} = -\frac{s-1}{r^{s+1}}\,v,\quad \frac{dQ}{d\xi_1} = -\frac{s-1}{r^{s+1}}\,v_1,\ \ \&c.$$

The integrations in equation (3) may be extended to all the values of the variables which satisfy the condition

$$v^2 + v_1^2 + v_2^2 + . . + v_s^2 \ \overline{\overline{<}}\ a^2;$$

and the limits in (4) will then be such as to include all the values which satisfy the equation

$$v^2 + v_1^2 + v_2^2 + . . + v_s^2 = a^2,\quad \text{or}\quad r^2 = a^2,\ \&c.$$

Hence we have in the successive terms the second member of (4),

$$\frac{dQ}{dv} = -\frac{s-1}{a^{s+1}}\,v,\quad \frac{dQ}{dv_1} = -\frac{s-1}{a^{s+1}}\,v_1,\ \ \&c.$$

If in the integrations we only take the positive values of the variables v, v_1, v_2, &c. which satisfy the limiting condition, we must multiply each integral by 2^{s+1}. Thus we have

$$uU = \frac{2^s P}{a^{s+1}}\left(\iint . . v\,dv_1\,dv_2 . . dv_s + \iint . . v_1\,dv\,dv_2 . . dv_s + \&c.\right)$$

$$= \frac{2^s (s+1) P}{a^{s+1}} \iint . . (a^2 - v_1^2 - v_2^2 - . . . - v_s^2)^{\frac{1}{2}}\,dv_1\,dv_2 . . dv_s$$

$$= (s+1) P \iint . . (1 - l_1 - l_2 . . - l_s)^{\frac{1}{2}}\,l_1^{-\frac{1}{2}}\,l_2^{-\frac{1}{2}} . . l_s^{-\frac{1}{2}}\,dl_1\,dl_2 . . dl_s;$$

in which last expression the limits include all positive values satisfying the condition

$$l_1 + l_2 + . . + l_s \ \overline{\overline{<}}\ 1,$$

Hence, by Liouville's theorem,

$$uU = (s+1)P\frac{\Gamma(\frac{1}{2})^s}{\Gamma\frac{1}{2}(s+1)}\cdot\tfrac{1}{2}s\int_0^1(1-h)^{\frac{1}{2}}h^{\frac{1}{2}(s-1)}dh = \frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)}P,$$

which gives the required value of the integral U.

If we denote by U' any integral corresponding to U, in which the system of variables $u, x_1, x_2 \ldots$ and $u', x_1', x_2' \ldots$ are inverted, we shall have $uU = u'U'$, since P is a function symmetrical with respect to the two systems; and we therefore deduce from the preceding result,

$$\left.\begin{aligned} & u\left[\int_{-\infty}^{\infty}\right]^s\frac{[d\xi]^s}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s+1)}\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s-1)}} \\ = & u'\left[\int_{-\infty}^{\infty}\right]^s\frac{[d\xi]^s}{\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s+1)}\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s-1)}} \\ = & \frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)}\;\frac{1}{\{\Sigma(x-x')^2+(u+u')^2\}^{\frac{1}{2}(s-1)}} \end{aligned}\right\}\ldots\ldots(5).$$

I shall add another demonstration of this theorem, as an application of some remarkable analysis given by Mr. Green in his memoir "On the determination of the exterior and interior attractions of ellipsoids of variable densities."*

$$\text{Let } V = \left[\int_{-\infty}^{\infty}\right]^s\frac{u\,[d\xi]^s}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s+1)}\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}\ldots(6),$$

an integral which may also be expressed thus:

$$-\frac{1}{n-1}\frac{d}{du}\left\{\left[\int_{-\infty}^{\infty}\right]^s\frac{[d\xi]^s}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s-1)}\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}\right\}.$$

From this latter form, we see that the equation

$$\frac{d^2V}{du^2}+\Sigma\frac{d^2V}{dx^2} = 0\ldots\ldots\ldots\ldots\ldots(7)$$

is satisfied, provided u does not vanish. Hence V is a function which satisfies this equation for all values of $x_1, x_2 \ldots$ and for all the values of u between 0 and ∞. At these limits the value of V may be easily determined, and the general value inferred in the following manner.

When $u = 0$ the quantity under the signs of integration in the expression for V vanishes for all the values of $\xi_1, \xi_2 \ldots$ which are not equal to $x_1, x_2 \ldots$ respectively. Hence it follows that, when $u = 0$,

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* Read at the Cambridge Philosophical Society, May 6, 1846. See *Trans.*, vol. v.

value as V, and therefore, by a theorem of Green's, in the memoir referred to, must be equal to V for all positive values of u.

From what has been proved above we may deduce the solution of the following problem:

Having given for all values of ξ_1, ξ_2 ...., the value of the multiple integral

$$\mathrm{S}\,\frac{\rho'\,dx_1'\,dx_2'\,..\,dx_s'}{\{(x_1'-\xi_1)^2+(x_2'-\xi_2)^2+..+(x_s'-\xi_s)^2+u'^2\}^{\frac{1}{2}(s+1)}}\;....(a),$$

where u' and ρ' are any functions of x_1', x_2'....x_s', let it be required to find the value of

$$\mathrm{S}\,\frac{\rho'\,dx_1'\,dx_2'\,..\,dx_s'}{\{(x_1'-x_1)^2+(x_2'-x_2)^2+...+(x_s'-x_s)^2+(u'+u)^2\}^{\frac{1}{2}(s+1)}}\;..(b),$$

where x_1, x_2.., x_s are any given quantities, and u a given positive quantity.

Denoting the expression (a) by Φ, and the expression (b) by ϕ, we have, from the theorem established above,

$$\phi = \frac{u\Gamma \frac{1}{2}(s+1)}{\pi^{\frac{1}{2}(s+1)}} \mathop{\mathrm{S}} \rho'\, dx_1'\, dx_2' \ldots dx_s'.$$

$$\left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s}{\{\Sigma(\xi - x)^2 + u^2\}^{\frac{1}{2}(s+1)} \{\Sigma(\xi - x')^2 + u'^2\}^{\frac{1}{2}(s-1)}}$$

$$= \frac{u\Gamma \frac{1}{2}(s+1)}{\pi^{\frac{1}{2}(s+1)}} \left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s}{\{\Sigma(\xi - x)^2 + u^2\}^{\frac{1}{2}(s+1)}} \cdot \mathop{\mathrm{S}} \frac{\rho'\, dx_1'\, dx_2' \ldots dx_s}{\{\Sigma(\xi - x')^2 + u'^2\}^{\frac{1}{2}(s-1)}},$$

or
$$\phi = \frac{u\Gamma \frac{1}{2}(s+1)}{\pi^{\frac{1}{2}(s+1)}} \left[\int_{-\infty}^{\infty}\right]^s \frac{\Phi\, [d\xi]^s}{\{\Sigma(\xi - x)^2 + u^2\}^{\frac{1}{2}(s+1)}} \ldots\ldots (c).$$

But, by hypothesis, ϕ' is given for all values of $\xi_1, \xi_2 \ldots \xi_s$; and therefore this equation expresses the solution of the problem. We may also deduce from the theorem (5) the expression

$$\phi = -\frac{\Gamma(\frac{1}{2}s + 1)}{(s-1)\pi^{\frac{1}{2}(s+1)}} \left[\int_{-\infty}^{\infty}\right]^s \frac{\Psi\, [d\xi]^s}{\{\Sigma(\xi - x)^2 + u^2\}^{\frac{1}{2}(s-1)}} \ldots (d),$$

by means of which ϕ may be determined when the value, Ψ, of $\frac{d\phi}{du}$ corresponding to $u = 0$ is given.

For the particular case of $u' = 0$, the theorem (d) is included in a theorem given by Green, in which the number n in the exponent of the denominator may differ from the number s of variables, the sole condition being that $n - s + 1$ must be positive; but it is only in the case of $n = s$ that a general theorem such as (d), by means of which the general value of ϕ is obtained from the value $\frac{d\phi}{du}$ when $u = 0$, is obtained, can be established.

Let us now apply these formulæ to the case of $x = 2$: we may in this case conveniently replace x_1, x_2, u by x, y, z, and ξ_1, ξ_2, by ξ, η. Equations (c) and (d) become

$$\phi = \frac{z}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\Phi\, d\xi\, d\eta}{\{(\xi - x)^2 + (\eta - y)^2 + z^2\}^{\frac{3}{2}}} \ldots\ldots (e),$$

and
$$\phi = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\Psi\, d\xi\, d\eta}{\{(\xi - x)^2 + (\eta - y)^2 + z^2\}^{\frac{1}{2}}} \ldots (f),$$

where Ψ denotes the value of $\frac{d\phi}{dz}$ when $x = \xi$, $y = \eta$, $z = 0$.

The first of these theorems may be deduced from a very general theorem given by Green in his essay on Electricity and Magnetism. The second may be demonstrated in the following manner.

between two lines parallel to OY and at equal distances, a, on its two sides, has a constant value c, and the temperature of the remainder of the plane zero. In this case the formula (e) will give, for the temperature at a point (x, y, z) above the plane,

$$\begin{aligned}\phi &= \frac{zc}{2\pi}\int_{-\infty}^{\infty}\int_{-a}^{a}\frac{d\xi\, d\eta}{\{(\xi - x)^2 + (\eta - y)^2 + z^2\}^{\frac{3}{2}}}\\ &= \frac{c}{\pi}\left(\tan^{-1}\frac{x+a}{z} - \tan^{-1}\frac{x-a}{z}\right)\\ &= \frac{c}{\pi}\tan^{-1}\frac{2az}{x^2+z^2-a^2}.\end{aligned}$$

From this we conclude that the isothermal surfaces which correspond to this case are circular cylinders, which intersect the plane (xy) in the two parallel lines bounding A.

The application to this example, and all others in which the isothermal surfaces are cylindrical, may be made directly by putting $s = 1$ in the general formulæ.

II.

I now proceed to find the values, which will be denoted by V and W, of the integrals

$$\left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s\,[\cos m\xi]^s}{(\Sigma\xi^2 + u^2)^{\frac{1}{2}(s+1)}}$$

and $$\left[\int_{-\infty}^{\infty}\right]^s [dm]^s\,[\cos mx]^s\,\frac{\epsilon^{-(\Sigma m^2)^{\frac{1}{2}}u}}{(\Sigma m^2)^{\frac{1}{2}}},$$

where the symbols $[\cos m\xi]^s$, $[\cos mx]^s$ denote the products

$$\cos m_1\xi_1 \,.\, \cos m_2\xi_2 \ldots \cos m_s\xi_s,$$

$$\cos m_1x_1 \,.\, \cos m_2x_2 \ldots \cos m_sx_s;$$

and the notation is in other respects the same as before.

By means of the formula

$$[\cos m\xi + \sin m\xi\,.\,\sqrt{(-1)}]^s = \cos(\Sigma m\xi) + \sin(\Sigma m\xi)\,.\,\sqrt{(-1)},$$

it is easily shewn that

$$V = \left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s \cos\Sigma(m\xi)}{(\Sigma\xi^2 + u^2)^{\frac{1}{2}(s+1)}} \ \ldots\ldots\ldots\ (a).$$

Hence, by a suitable linear transformation, in which one of the assumptions is $\Sigma m\xi = \eta(\Sigma m^2)^{\frac{1}{2}}$, we have

$$V = \int_{-\infty}^{\infty}\cos\mu\eta\,.\,d\eta\,.\left[\int_{-\infty}^{\infty}\right]^{s-1}\frac{[d\zeta]^{s-1}}{(u^2 + \eta^2 + \Sigma\zeta^2)^{\frac{1}{2}(s+1)}}\ \ldots(b).$$

Now, by means of Liouvil

$$\left[\int_{-\infty}^{\infty}\right]^{s-1} \frac{[d\zeta]^{s-1}}{(u^2+\eta^2+\Sigma\zeta^2)^{\frac{1}{2}(s-1)}} =$$

Hence $\quad V = \dfrac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\displaystyle\int_0$

Differentiating with respect
duction of the integral will t

$$-\frac{dV}{du} = (s-1)\frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\int_0^\infty\int_0^\infty \frac{u\,.\,\xi^{s-2}\cos\mu\eta\,.\,d\xi\,d\eta}{(\xi^2+\eta^2+u^2)^{\frac{1}{2}(s+1)}} \,..(d).$$

Now

$$\int_0^\infty \frac{\xi^{s-2}\,d\xi}{(\xi^2+\eta^2+u^2)^{\frac{1}{2}(s+1)}} = \int_0^\infty \frac{\xi^{-3}\,d\xi}{\left(1+\dfrac{\eta^2+u^2}{\xi^2}\right)^{\frac{1}{2}(s+1)}} = \frac{1}{2}\int_0^\infty \frac{dt}{\{1+(\eta^2+u^2)t\}^{\frac{1}{2}(s+1)}}$$

$$= \frac{1}{s-1}\cdot\frac{1}{\eta^2+u^2}.$$

Hence
$$-\frac{dV}{du} = \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\int_0^\infty \frac{u\cos\mu\eta\,d\eta}{\eta^2+u^2}\,\ldots\ldots\ldots(e),$$

$$= \frac{2\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s-1)}\cdot\epsilon^{-\mu u}.$$

From this, by integration with respect to u, we deduce the value of V: thus we have the result

$$\left[\int_{-\infty}^{\infty}\right]^{s}\frac{[d\xi]^s\,[\cos m\xi]^s}{(\Sigma\xi^2+u^2)^{\frac{1}{2}(s-1)}} = \frac{2\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s-1)}\,\frac{\epsilon^{-(\Sigma m^2)^{\frac{1}{2}}u}}{(\Sigma m^2)^{\frac{1}{2}}}\,\ldots(V).$$

To evaluate the integral W we may in the first place reduce it to a double integral by a process similar to that indicated above, for obtaining the expression (c); and we thus find

$$W = \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\int_0^\infty\int_0^\infty \frac{dm\,dn\,.\,m^{s-2}\cos(nr)\,.\,\epsilon^{-(m^2+n^2)^{\frac{1}{2}}u}}{(m^2+n^2)^{\frac{1}{2}}}\,..(a),$$

where r denotes $(\Sigma x^2)^{\frac{1}{2}}$. If we take $m=\rho\cos\vartheta$, $n=\rho\sin\vartheta$, this becomes

$$W = \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\int_0^\infty\int_0^{\frac{1}{2}\pi} d\theta\,d\rho\,\rho^{s-2}\cos^{s-2}\vartheta\cos(r\rho\sin\vartheta)\,\epsilon^{-\rho u}\,.\,.(b).$$

Now we have

$$\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)\cos(r\rho\sin\vartheta)\,\epsilon^{-\rho u} = \rho^{2}\cos^{2}\vartheta\,.\cos(r\rho\cos\vartheta)\epsilon^{-\rho u}\,.\,.(c).$$

* See vol. II. p. 221, First Series.

Considering first the case where s is even, let $f = \frac{1}{2}s - 1$; we thus find

$$\rho^{s-2}\cos^{s-2}\vartheta\cos(r\rho\cos\vartheta)\,\epsilon^{-\rho u} = \left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}s-1}\cos(r\rho\sin\vartheta)\,\epsilon^{-\rho u},$$

and, by substitution in (b), we have

$$W = \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\int_0^\infty\int_0^{\frac{1}{2}\pi} d\vartheta\, d\rho\,.\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}s-1}\cos(r\rho\sin\vartheta)\,\epsilon^{-\rho u}$$

$$= \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}s-1}\int_0^\infty\int_0^{\frac{1}{2}\pi} d\vartheta\, d\rho\cos(r\rho\sin\vartheta)\,\epsilon^{-\rho u}$$

$$= \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}s-1}\int_0^{\frac{1}{2}\pi}\frac{u\,d\vartheta}{u^2+r^2\sin^2\vartheta}$$

$$= \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}s-1}\frac{\frac{1}{2}\pi}{(u^2+r^2)^{\frac{1}{2}}}$$

$$= \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\,\frac{1.3^2\ldots(s-1)^2}{(u^2+r^2)^{\frac{1}{2}(s-1)}} = 2^{s-1}\pi^{\frac{1}{2}(s-1)}\Gamma\tfrac{1}{2}(s-1)\frac{1}{(u^2+r^2)^{\frac{1}{2}(s-1)}}$$
$$\ldots\ldots(d).$$

In the second case, when s is odd, let $f = \frac{1}{2}(s-1)$ in (c); then, making use of the result in (b), we have

$$W = \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}(s-1)}$$
$$\int_0^\infty\int_0^{\frac{1}{2}\pi} d\vartheta\, d\rho\,\rho\cos\vartheta\,.\cos(r\rho\sin\vartheta)\,\epsilon^{-\rho u}$$

$$= \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}(s-1)}\int_0^\infty d\rho\,.\frac{\sin(r\rho)}{r}\,\epsilon^{-\rho u}$$

$$= \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}(s-1)}\frac{1}{r^2+\rho^2}$$

$$= 2^{s-1}\pi^{\frac{1}{2}(s-1)}\Gamma\tfrac{1}{2}(s-1)\frac{1}{(u^2+r^2)^{\frac{1}{2}(s-1)}}\,.$$

Hence, whether s be odd or even, we conclude that

$$\left[\int_{-\infty}^{\infty}\right]^s [dm]^s\,[\cos mx]^s\,\frac{\epsilon^{-(\Sigma m^2)^{\frac{1}{2}}u}}{(\Sigma m^2)^{\frac{1}{2}}}$$
$$= 2^{s-1}\pi^{\frac{1}{2}(s-1)}\Gamma\tfrac{1}{2}(s-1)\frac{1}{(u^2+\Sigma x^2)^{\frac{1}{2}(s-1)}}\ \ldots\ldots(W).$$

The investigation which we have just gone through, of the integrals (V), (W) constitutes the verification of "Fourier's theorem" in a particular case. For, by this theorem, we

have, if $F(x_1, x_2 ..)$ be a function which remains the same when the signs of any of the variables are changed,

$$2^s\pi^s F(x_1, x_2 ..)$$

$$= \left[\int_{-\infty}^{\infty}\right]^s [dm]^s [\cos mx]^s \int_{-\infty}^{\infty} [d\xi]^s [\cos m\xi]^s F(\xi_1, \xi_2 ..) \; .. \; (e);$$

and if we take

$$F(\xi_1, \xi_2 ..) = \frac{1}{(\Sigma\xi^2 + u^2)^{\frac{1}{2}(s-1)}},$$

the result of the integrations with respect to $\xi_1, \xi_2 ..$, is given by (V), and the second member thus becomes a multiple integral with respect to $m_1, m_2 ..$, which is shewn by (W) to be equal to the first member. Conversely, if we assume Fourier's theorem, we may deduce the value W, by means of it, from that of V. The integrals V and W are also connected by means of another case of Fourier's theorem, found by taking, in (e),

$$F(\xi_1, \xi_2 ..) = \frac{\epsilon^{-(\Sigma\xi^2)^{\frac{1}{2}}u}}{(\Sigma\xi^2)^{\frac{1}{2}}}.$$

In this way, after the value of W has been found, that of V may be deduced.

The formulæ (V) and (W) may be applied to evaluate the multiple integral u, and we shall thus obtain the result of the investigation in §. I. in a different manner.

By means of the equation obtained by differentiating (W) with respect to u, we find

$$\frac{u}{\{\Sigma(\xi - x)^2 + u^2\}^{\frac{1}{2}(s+1)}} = \frac{1}{2^{s-1}(s-1)\pi^{\frac{1}{2}(s-1)}\Gamma\frac{1}{2}(s-1)}$$

$$\left[\int_{-\infty}^{\infty}\right]^s [dm]^s [\cos m(\xi - x)]^s \epsilon^{-(\Sigma m^2)^{\frac{1}{2}}u};$$

Making this substitution, for one of the factors of the expression under the integral signs in U, we have

$$Uu = \frac{1}{2^{s-1}(s-1)\pi^{\frac{1}{2}(s-1)}\Gamma\frac{1}{2}(s-1)} \left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s}{\{\Sigma(\xi - x')^2 + u'^2\}^{\frac{1}{2}(s-1)}}$$

$$\left[\int_{-\infty}^{\infty}\right] [dm]^s [\cos m(\xi - x)]^s \epsilon^{-(\Sigma m^2)^{\frac{1}{2}}u}$$

$$= \frac{1}{2^{s-1}(s-1)\pi^{\frac{1}{2}(s-1)}\Gamma(\frac{1}{2}s - 1)}$$

$$\left[\int_{-\infty}^{\infty}\right]^s [dm]^s [\cos m(x - x')]^s \epsilon^{-(\Sigma m^2)^{\frac{1}{2}}u} \left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s [\cos m(\xi - x')]^s}{\{\Sigma(\xi - x')^2 + u'^2\}^{\frac{1}{2}(s-1)}}$$

$$= \frac{\pi}{2^{s-2}(s-1)\{\Gamma\tfrac{1}{2}(s-1)\}^2}$$

$$\left[\int_{-\infty}^{\infty}\right]^s [dm]^s [\cos m(x-x')]^s \, \varepsilon^{-(\Sigma m^2)^{\frac{1}{2}}u} \frac{\varepsilon^{-(\Sigma m^2)^{\frac{1}{2}}u'}}{(\Sigma m^2)^{\frac{1}{2}}}, \text{ by } (V),$$

$$= \frac{2\pi^{\frac{1}{2}(s+1)}}{(s-1)\,\Gamma\tfrac{1}{2}(s-1)} \frac{1}{\{\Sigma(x-x')^2+(u+u')^2\}^{\frac{1}{2}(s-1)}}, \text{ by } (W),$$

which agrees with the value obtained above.

III.

The value of the integral U may also be obtained by a direct process of reduction, as follows.

By a suitable linear transformation, in which assumptions such as

$$\xi_1 - x_1 = \Sigma a\zeta$$

are made, we find

$$U = \left[\int_{-\infty}^{\infty}\right] \frac{[d\zeta]^s}{(\Sigma\zeta^2+u^2)^{\frac{1}{2}(s+1)}(\Sigma\zeta^2-2f\zeta_1+f^2+u'^2)^{\frac{1}{2}(s-1)}} \dots (a),$$

where

$$f^2 = \Sigma(x-x')^2.$$

Let us now assume

$$\zeta_1 = \rho\cos\phi, \quad \zeta_2 = \rho\sin\phi\cos\theta_1, \quad \zeta_3 = \rho\sin\phi\sin\theta_1\cos\theta_2 \dots,$$

$$\zeta_{s-1} = \rho\sin\phi\sin\theta_1\sin\theta_2\dots\cos\theta_{s-2},$$

$$\zeta_s = \rho\sin\phi\sin\theta_1\sin\theta_2\dots\sin\theta_{s-2},$$

from which we deduce*

$$[d\xi]^s = \rho^{s-1}\sin^{s-2}\phi\,\sin^{s-3}\theta_1\,\sin^{s-4}\theta_2\dots\sin\theta_{s-3}[d\theta]^{s-2}\,d\phi\,d\vartheta;$$

a transformation given first by Green. Equation (a) is thus reduced to

$$U = H_{s-2}\int_0^{\infty}\int_0^{2\pi} \frac{\rho^{s-1}\sin^{s-2}\phi\,d\phi\,d\rho}{(\rho^2+u^2)^{\frac{1}{2}(s+1)}(\rho^2-2\rho f\cos\phi+f^2+u'^2)^{\frac{1}{2}(s-1)}} \dots (b),$$

where H_{s-2} denotes the product

$$\int_0^{\pi}\sin^{s-3}\theta\,d\theta \,.\int_0^{\pi}\sin^{s-4}\theta\,d\theta\dots\int_0^{\pi}d\theta.$$

Let $\rho = u\tan\tfrac{1}{2}\vartheta$; we thus get

$$Uu = \tfrac{1}{2}H_{s-2}\int_0^{\pi}\int_0^{2\pi}$$

$$\frac{\sin^{s-1}\vartheta\,\sin^{s-2}\phi\,d\phi\,d\vartheta}{\{2(f^2+u'^2+u^2)+2(f^2+u'^2-u^2)\cos\vartheta-4uf\sin\vartheta\cos\phi\}^{\frac{1}{2}(s-1)}},$$

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* See vol. IV. p. 24., First Series.

and we may now conveniently assume

$$2(f^2 + u'^2 + u^2) = h^2 + k^2,$$

$$2(f^2 + u'^2 - u^2)\cos\vartheta - 4uf\sin\vartheta\cos\phi$$

$$= 2\{(f^2 + u^2 - u'^2)^2 + 4u^2f^2\}^{\frac{1}{2}}\cos\theta = 2hk\cos\theta,$$

and $$\sin\phi\sin\vartheta = \sin\phi\sin\theta,$$

from which we deduce

$$h^2 = (u' + u)^2 + f^2, \quad k^2 = (u' - u)^2 + f^2,$$

$$\sin\vartheta\, d\phi\, d\vartheta = \sin\theta\, d\varphi\, d\theta;$$

the expression for U becomes

$$Uu = \tfrac{1}{2}H_{s-2}\int_0^{\pi}\int_0^{2\pi}\frac{\sin^{s-1}\theta\sin^{s-3}\varphi\, d\varphi\, d\theta}{(h^2-2hk\cos\theta-k^2)^{\frac{1}{2}(s-1)}}$$

$$= H_{s-1}\int_0^{\pi}\frac{\sin^{s-1}\theta\, d\theta}{(h^2 - 2hk\cos\theta + k^2)^{\frac{1}{2}(s-1)}}.$$

Let $$h\sin(\psi - \theta) = k\sin\psi;$$

by means of this transformation, observing that $h > k$, we readily find

$$Uu = \frac{H_s}{h^{s-1}}$$

or $$Uu = \frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)}\frac{1}{\{\Sigma(x-x')^2 + (u+u')^2\}^{\frac{1}{2}(s-1)}},$$

which is the same as the result previously obtained.

St. Peter's College, Oct. 3, 1846.

ON CERTAIN FORMULÆ FOR DIFFERENTIATION, WITH APPLICATIONS TO THE EVALUATION OF DEFINITE INTEGRALS.

By Arthur Cayley.

In attempting to investigate a formula in the theory of multiple definite integrals (which will be noticed in the sequel), I was led to the question of determining the $(i+1)^{\text{th}}$ differential coefficient of the $2i^{\text{th}}$ power of $\sqrt{(x+\lambda)} - \sqrt{(x+\mu)}$; the only way that occurred for effecting this was to find the successive differential coefficients of this quantity, which may be effected as follows. Assume

$$U_{k,i} = \{(x+\lambda)(x+\mu)\}^{\frac{1}{2}k}\{\sqrt{(x+\lambda)} - \sqrt{(x+\mu)}\}^{2i},$$

then

$$\frac{1}{U_{k,i}}\frac{d}{dx}U_{k,i}=\tfrac{1}{2}k\,\frac{2x+\lambda+\mu}{(x+\lambda)(x+\mu)}-\frac{i}{\surd\{(x+\lambda)(x+\mu)\}}$$

$$=\tfrac{1}{2}k\,\frac{\{\surd(x+\lambda)+\surd(x+\mu)\}^2-2\surd\{(x+\lambda)(x+\mu)\}}{(x+\lambda)(x+\mu)}-\frac{i}{\surd\{(x+\lambda)(x+\mu)\}}$$

$$=\tfrac{1}{2}k\,\frac{(\lambda-\mu)^2}{\{\surd(x+\lambda)-\surd(x+\mu)\}^2(x+\lambda)(x+\mu)}-\frac{k+i}{\surd\{(x+\lambda)(x+\mu)\}}\,.$$

Or, attending to the signification of $U_{k',i}$,

$$\frac{d}{dx}U_{k,i}=\tfrac{1}{2}k(\lambda-\mu)^2\,U_{k-2,i-1}-(k+i)\,U_{k-1,i}.$$

Hence $-\frac{1}{i}\frac{d}{dx}U_{0,i}=U_{-1,i}$

$$\frac{1}{i}\frac{d^2}{dx^2}U_{0,i}=\tfrac{1}{2}(\lambda-\mu)^2\,U_{-3,i-1}+(i-1)\,U_{-2,i},$$

&c.

from which the law is easily seen to be of the form

$$\frac{(-)^r}{i}\left(\frac{d}{dx}\right)^r U_{0,i}=S_\theta\,K_{r,\theta}(\lambda-\mu)^{2r-2-2\theta}\,U_{-2r+1+\theta,\,i-r+1+\theta}$$

(where the extreme values of θ are 0 and $(r-1)$ respectively) and $K_{r,\theta}$ is determined by

$$K_{r+1,\theta+1}=(r-1-\tfrac{1}{2}\theta)\,K_{r,\theta+1}+(i-3r+2+2\theta)\,K_{r,\theta}.$$

This equation is satisfied by

$$K_{r,\theta}=\frac{\Gamma(r-\frac{1}{2}-\theta)\,\Gamma(2r-1-\theta)\,\Gamma(i-r+\theta+1)}{\Gamma\frac{1}{2}\,\Gamma(\theta+1)\,\Gamma(2r-1-2\theta)\,\Gamma(i-r+1)}\,.$$

For in the first place this gives

$$(r-1-\tfrac{1}{2}\theta)\,K_{r,\theta+1}$$

$$=\frac{(r-1-\frac{1}{2}\theta)}{\Gamma\frac{1}{2}\,\Gamma(\theta+2)}\;\frac{\Gamma(r-\frac{3}{2}-\theta)\,\Gamma(2r-2-\theta)\,\Gamma(i-r+\theta+2)}{\Gamma(2r-3-2\theta)\,\Gamma(i-r+1)}$$

$$=\frac{\Gamma(r-\frac{1}{2}-\theta)\,\Gamma(2r-1-\theta)\,\Gamma(i-r+\theta+2)}{\Gamma(\frac{1}{2})\,\Gamma(\theta+2)\,\Gamma(2r-2-2\theta)\,\Gamma(i-r+1)}\,.$$

And hence the second side of the equation reduces itself to

$$\frac{\Gamma(r-\frac{1}{2}-\theta)\,\Gamma(2r-1-\theta)\,\Gamma(i-r+\theta+1)}{\Gamma(\frac{1}{2})\,\Gamma(\theta+2)\,\Gamma(2r-1-2\theta)\,\Gamma(i-r+1)}$$

$$\{2(r-1-\theta)(i-r+\theta+1)+(\theta+1)(i-3r+2-2\theta)\},$$

this gives

$$\int_0^\infty \frac{x^{i-\frac{1}{2}}dx}{\{(x+\lambda)(x+\mu)\}^i} = \frac{\Gamma\frac{1}{2}\,\Gamma(i-\frac{1}{2})}{\Gamma i}\,\frac{1}{(\sqrt{\lambda}+\sqrt{\mu})^{2i-1}} \;\ldots (4),^*$$

whence also

$$\int_0^\infty \frac{x^{i-\frac{1}{2}}dx}{(x+\lambda)^{i+1}(x+\mu)^i} = \frac{\Gamma\frac{1}{2}\,\Gamma(i+\frac{1}{2})}{\Gamma i}\,\frac{1}{(\sqrt{\lambda}+\sqrt{\mu})^{2i}\sqrt{\lambda}} \;\ldots (5).$$

And from these, by simple transformations,

$$\int_\beta^\alpha \frac{(\alpha-x)^{i-\frac{1}{2}}(x-\beta)^{i-\frac{1}{2}}dx}{\{(\alpha-x)+m(x-\beta)\}^i} = \frac{\Gamma\frac{1}{2}\,\Gamma(i+\frac{1}{2})}{\Gamma(i+1)}\,\frac{(\alpha-\beta)^i}{(\sqrt{m}+1)^{2i}} \;\ldots (6),$$

$$\int_\beta^\alpha \frac{(\alpha-x)^{i-\frac{1}{2}}(x-\beta)^{i-\frac{3}{2}}dx}{\{(\alpha-x)+m(x-\beta)\}^i} = \frac{\Gamma\frac{1}{2}\,\Gamma(i-\frac{1}{2})}{\Gamma i}\,\frac{(\alpha-\beta)^{i-1}}{(\sqrt{m}+1)^{2i-1}} \ldots (7).$$

These last two formulæ are connected also by the following general property:

"If $\qquad (a, b, i) = \displaystyle\int_\beta^\alpha \frac{(\alpha-x)^{a-1}(x-\beta)^{b-1}dx}{\{(\alpha-x)+m(x-\beta)\}^i},$

then $(a, b, i) = \dfrac{\Gamma a \Gamma b}{\Gamma(a+b-i)\,\Gamma i}(\alpha-\beta)^{b-i}(a+b-i,\, i,\, b)$"...(8),

which I have proved by means of a double integral. From (6) we may obtain for $\gamma < 1$,

$$\int_{-1}^1 \frac{(1-x^2)^{i-\frac{1}{2}}dx}{(1-2\gamma x+\gamma^2)^i} = \frac{\Gamma\frac{1}{2}\,\Gamma(i+\frac{1}{2})}{\Gamma(i+1)} \;\ldots\ldots\ldots\ldots (9),$$

which however is only a particular case of

$$\int_{-1}^1 dx\,(1-x^2)^{i-\frac{1}{2}}(1-2\gamma x+\gamma^2)^{-i}\frac{d}{d\beta}\left[\beta^i\left(1-2\frac{\beta}{\gamma}x+\frac{\beta^2}{\gamma^2}\right)^{-i}\right]$$

$$= \frac{\Gamma\frac{1}{2}\,\Gamma(i+\frac{1}{2})}{\Gamma(i+1)}\,\beta^{i-1}(1-\beta)^{-2i} \ldots\ldots (10),$$

which supposes γ and $\dfrac{\beta}{\gamma}$ each less than unity. This formula was obtained in the case of $(i+\frac{1}{2})$ an integer, from a theorem, *Leg. Cal. Int.*, tom. II. p. 258, but there is no doubt that it is generally true.

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* This is immediately transformed into

$$\int_0^\infty \frac{x^{i-\frac{1}{2}}dx}{(ax^2+bx+c)^i} = \frac{\Gamma\frac{1}{2}\,\Gamma(i-\frac{1}{2})}{\Gamma i}\,\frac{1}{\{b+2\sqrt{(ac)}\}^{i-\frac{1}{2}}},$$

which is a particular case of a formula which will be demonstrated in a subsequent paper.

where σ extends from 0 to λ. Hence substituting, and prefixing the summatory sign

$$W = \pi^{i+\frac{1}{2}} S \frac{(-)^{\lambda} \Delta^{\lambda}}{2^{2\lambda}\, \Gamma(\lambda+1)\, \Gamma(i+\lambda+\frac{1}{2})} \left(\frac{d}{du}\right)^{2\lambda} \int_0 \xi^{i-\frac{1}{2}} (\xi + u^2)\, d\xi,$$

where λ extends from 0 to ∞, the formula required.

ON THE CAUSTIC BY REFLECTION AT A CIRCLE.

By Arthur Cayley.

The following solution of the problem is that given by M. de St. Laurent (*Annales de Gergonne*, tom. XVII. p. 128); the process of elimination is somewhat different.

The centre of the circle being taken for the origin, let k be its radius; a, b the coordinates of the luminous point; ξ, η those of the point at which the reflection takes place; x, y of any point in the reflected ray: we have in the first place

$$\xi^2 + \eta^2 = k^2 \ldots\ldots\ldots\ldots\ldots (1).$$

There is no difficulty in finding the equation of the reflected ray*

$$(b\xi - a\eta)(\xi x + \eta y - k^2) + (y\xi - \xi\eta)(a\xi + b\eta - k^2) = 0.$$

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* To do this in the simplest way, write

$$\rho^2 = (\xi - x)^2 + (\eta - y)^2, \quad \sigma^2 = (\xi - a)^2 + (\eta - b)^2.$$

Then, by the condition of reflection,

$$\rho + \sigma = \text{min.},$$

ρ, σ being considered as functions of the variables ξ, η, which are connected by the equation (1). Hence

$$\frac{\xi - x}{\rho} + \frac{\xi - a}{\sigma} + \lambda\xi = 0,$$

$$\frac{\eta - y}{\rho} + \frac{\eta - b}{\sigma} + \lambda\eta = 0.$$

Or eliminating λ,

$$\frac{\eta x - \xi y}{\rho} + \frac{\eta a - \xi b}{\sigma} = 0,$$

whence

$$(\eta x - \xi y)^2 [(\xi - a)^2 + (\eta - b)^2] = (\eta a - \xi b)^2 [(\xi - x)^2 + (\eta - y)^2].$$

Or

$$\{(\eta x - \xi y)(\xi - a) - (\eta a - \xi b)(\xi - x)\} [(\eta x - \xi y)(\xi - a) + (\eta a - \xi b)(\xi - x)]$$
$$+ \{(\eta x - \xi y)(\eta - b) - (\eta a - \xi b)(\eta - y)\} [(\eta x - \xi y)(\eta - b) + (\eta a - \xi b)(\eta - y)] = 0.$$

The factors in { } reduce themselves respectively to ξP and ηP, where $P = \xi(b - y) - \eta(a - x) + ay - bx$, omitting the factor P, (which equated to zero, is the equation of the line through (a, b) and (ξ, η).) And replacing $\xi(\xi - a) + \eta(\eta - b)$ and $\xi(\xi - x) + \eta(\eta - y)$ by $k^2 - a\xi - b\eta$ and $k^2 - \xi x - \eta y$, respectively, we have the equation given above.

Hence, after a slight reduction,

$$V = \frac{\pi^{i+1}}{v\Gamma(i+1)} S \frac{(-)^\lambda\, \Gamma(i+\lambda+1)}{\Gamma(i+1)\,\Gamma(\lambda+1)} \frac{A^\lambda}{\{(u+v)^2\}^\lambda};$$

or finally $$V = \frac{\pi^{i+1}}{\Gamma(i+1)} \frac{1}{v\{(u+v)^2 + A\}^i} \quad \ldots\ldots\ldots (16),$$

a remarkable formula, the discovery of which is due to Mr. Thomson. It only remains to prove the formula for W. Out of the variety of ways in which this may be accomplished, the following is a tolerably simple one. In the first place, by a linear transformation corresponding to that between two sets of rectangular axes, we have

$$W = \int \frac{dx\, dy\,..}{\{(x - \sqrt{A})^2 + y^2 .. + u^2\}^i};$$

or expanding in powers of A, and putting for shortness $R = x^2 + y^2 \ldots + u^2$, the general term of W is

$$(-)^\sigma A^\lambda \frac{\Gamma(i+\lambda+\sigma)}{\Gamma i\, \Gamma(\lambda-\sigma+1)\, \Gamma(2\sigma+1)}\, 2^{2\sigma} \int x^{2\sigma} R^{-i-\lambda-\sigma} dx\, dy\,..$$

the limits being as before $x^2 + y^2 + \ldots = \xi$. To effect the integrations, write $\sqrt{\xi}\sqrt{x}$, $\sqrt{\xi}\sqrt{y}$, &c. for $x, y \ldots$ So that the equation of the limits becomes $x + y + \ldots = 1$. Also restricting the integral to positive values, we shall multiply it by 2^{2i+1}. The integral thus becomes

$$\xi^{\sigma+i+\frac{1}{2}} \int x^{\sigma-\frac{1}{2}} y^{-\frac{1}{2}} .. \{\xi(x+y..) + u^2)^{-i-\lambda-\sigma} dx\, dy.$$

Equivalent to

$$\xi^{\sigma+i+\frac{1}{2}} \frac{\Gamma(\sigma+\frac{1}{2})\,\pi^i}{\Gamma(i+\sigma+\frac{1}{2})} \int_0^1 \theta^{i+\sigma-\frac{1}{2}} (\xi\theta + u^2)^{-i-\lambda-\sigma} d\theta;$$

i.e. to $$\frac{\Gamma(\sigma+\frac{1}{2})\,\pi^i}{\Gamma(i+\sigma+\frac{1}{2})} \int_0 \xi^{i+\sigma-\frac{1}{2}} (\xi + u^2)^{-i-\lambda-\sigma} d\xi.$$

Or after a slight reduction, the general term of W is

$$\frac{\pi^{i+\frac{1}{2}}}{\Gamma i} (-)^{\lambda+\sigma} A^\lambda \frac{\Gamma(i+\lambda+\sigma)}{\Gamma(\sigma+1)\,\Gamma(\lambda-\sigma+1)\,\Gamma(i+\sigma+\frac{1}{2})} \int_0 \xi^{i+\sigma-\frac{1}{2}} (\xi+u^2)^{-i-\lambda-\sigma} d\xi,$$

where σ may be considered as extending from 0 to λ inclusively, and then λ from 0 to ∞. But by a formula easily proved

$$\left(\frac{d}{du}\right)^{2\lambda} (\xi + u^2)^{-i} = \frac{2^{2\lambda}\, \Gamma(\lambda+1)\, \Gamma(i+\lambda+\frac{1}{2})}{\Gamma i}$$

$$S(-)^\sigma \frac{\Gamma(i+\lambda+\sigma)}{\Gamma(\sigma+1)\,\Gamma(\lambda-\sigma+1)\,\Gamma(i+\sigma+\frac{1}{2})} \xi^\sigma (\xi + u^2)^{-i-\lambda-\sigma},$$

In (a) change x into $a-x$, and in (b) into $a+x$; these formulæ become

$$\phi(a-x) = (a-x)^D \phi(\epsilon^\theta), \quad \text{and} \quad \phi(a+x) = (a+x)^{-D} \phi(\epsilon^{-\theta});$$

whence we derive

$$\int_0^a \phi(a-x)\, x^{n-1} dx = \left\{ \int_0^a (a-x)^D x^{n-1} dx \right\} \phi(\epsilon^\theta)$$

$$= \frac{\Gamma(n)\,\Gamma(D+1)}{\Gamma(D+n+1)}\, a^{D+n} \phi(\epsilon^\theta) = \frac{\Gamma(n)\, a^{D+n}}{(D+1)(D+2)\dots(D+n)}\, \phi(\epsilon^\theta).$$

But
$$\int \phi(a)\, da = \frac{a^{D+1}}{D+1}\, \phi(\epsilon^\theta),$$

$$\int^2 \phi(a)\, da^2 = \frac{a^{D+2}}{(D+1)(D+2)}\, \phi(\epsilon^\theta),$$

$$\int^n \phi(a)\, da^n = \frac{a^{D+n}}{(D+1)(D+2)\dots(D+n)}\, \phi(\epsilon^\theta).$$

Therefore
$$\int_0^a \phi(a-x)\, x^{n-1} dx = \Gamma(n) \int^n \phi(a)\, da^n,$$

the second member to be integrated from $a = o$ to $a = a$, as the first vanishes when $a = o$.

$$\int_0^\infty \phi(a+x)\, x^{n-1} dx = \left\{ \int_0^\infty (a+x)^{-D} x^{n-1} dx \right\} \phi(\epsilon^{-\theta})$$

$$= \frac{\Gamma(n)\,\Gamma(D-n)}{\Gamma(D)}\, a^{-D+n} \phi(\epsilon^{-\theta}) = \frac{\Gamma(n)\, a^{-D+n}}{(D-1)(D-2)\dots(D-n)}\, \phi(\epsilon^{-\theta}).$$

$$\int \phi(a)\, da = -\frac{a^{-D+1}}{D-1}\, \phi(\epsilon^{-\theta}),$$

$$\int^2 \phi(a)\, da^2 = (-1)^2 \frac{a^{-D+2}}{(D-1)(D-2)}\, \phi(\epsilon^{-\theta}),$$

$$\int^n \phi(a)\, da^n = (-1)^n \frac{a^{-D+n}}{(D-1)(D-2)\dots(D-n)}\, \phi(\epsilon^{-\theta}).$$

Therefore
$$\int_0^\infty \phi(a+x)\, x^{n-1} dx = (-1)^n\, \Gamma(n) \int^n \phi(a)\, da^n,$$

the second member to be integrated from $a = A$ to $a = a$, A being the value of a which makes the first member vanish.

I cannot stop to comment on the several steps of these two examples, which are known,* and are only given here by

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* See vol. I. p. 114, First Series.

way of illustration. I now subjoin the integration of a differential equation, which may be more conveniently effected otherwise; but it may be well to shew that (*a*) may in some cases be thus applied.

Let $$x^2 \frac{d^2y}{dx^2} + mx\frac{dy}{dx} + ny = 0.$$

Make $y = \phi(x) = x^D\phi(\varepsilon^o)$. With this value of y, the proposed equation becomes

$$x^D \{D^2 + (m-1)D + n\}\phi(\varepsilon^o) = 0.$$

Or, as $\phi(\varepsilon^o)$ is independent of x,

$$\{D^2 + (m-1)D + n\}\phi(\varepsilon^o) = o,$$

the integral of which is $\phi(\varepsilon^o) = A\varepsilon^{b_1 o} + B\varepsilon^{b_2 o}$, b_1, b_2 being the roots of $b^2 + (m-1)b + n = o$. Therefore

$$y = \phi(x) = Ax^{b_1} + Bx^{b_2}.$$

In (*a*) change $\phi(x)$ into $\phi(\varepsilon^x)$; then $\phi(\varepsilon^x) = x^D\phi(\varepsilon^{\varepsilon^o})$. Change in this ε^x into x, or x into $\log x = lx$; we have

$$\phi(x) = (lx)^D\phi(\varepsilon^{\varepsilon^o}) \;\ldots\ldots\ldots\ldots\; (c).$$

This serves to develop $\phi(x)$ by the powers of $\log\log x = llx = l^2x$. We might apply it to integration. By a continued repetition of the same steps, we may find a formula to develop by the powers of l^nx.

We may treat $\phi(x) = (1+\Delta)^x\phi(o)$ in like manner; first changing $\phi(x)$ into $\phi(\varepsilon^x)$, then x into lx. Thus we should find

$$\left.\begin{aligned}\phi(x) &= (1+\Delta)^{lx}\phi(\varepsilon^o)\\ \phi(x) &= (1+\Delta)^{llx}\phi(\varepsilon^{\varepsilon^o})\end{aligned}\right\}\ldots\ldots\ldots\ldots(d).$$

The following is proved by developing the second member.

$$\frac{d^n\phi(x)}{dx^n} = \phi(x+D)\,o^n \ldots\ldots\ldots\ldots(e).$$

This gives $$\frac{d^n\phi(o)}{do^n} = \phi(D)\,o^n.$$

Therefore $$\phi(x) = \phi(o) + \frac{x}{1}\frac{d\phi(o)}{do} + \frac{x^2}{1.2}\frac{d^2\phi(o)}{do^2} + \&c.$$

$$= \phi(D)\left\{1 + \frac{ox}{1} + \frac{o^2x^2}{1.2} + \&c.\right\}.$$

Or $$\phi(x) = \phi(D)\,\varepsilon^{ox} \;\ldots\ldots\ldots\ldots\; (f).$$

This might be thus investigated:

$$D\varepsilon^{ox} = x\varepsilon^{ox} = x,\quad D^2\varepsilon^{ox} = x^2,\quad D^n\varepsilon^{ox} = x^n.$$

$$\tfrac{1}{2}\{\phi(\epsilon^{o\sqrt{(-1)}})+\phi(\epsilon^{-o\sqrt{(-1)}})\}=\phi(E)\cos ox,$$

$$\frac{1}{2\sqrt{(-1)}}\{\phi(\epsilon^{o\sqrt{(-1)}})-\phi(\epsilon^{-o\sqrt{(-1)}})\}=\phi(E)\sin ox,$$

$$\frac{1}{2}\int_0^\infty\frac{dx}{1+x^2}\{\phi(\epsilon^{o\sqrt{(-1)}})+\phi(\epsilon^{-o\sqrt{(-1)}})\}=\phi(E)\int_0^\infty\frac{dx\cos ox}{1+x^2}$$
$$=\tfrac{1}{2}\pi\,\phi(E)\,\epsilon^{-o}=\tfrac{1}{2}\pi\,\phi(\epsilon^{-1}),$$

$$\frac{1}{2\sqrt{(-1)}}\int_0^\infty\frac{x\,dx}{1+x^2}\{\phi(\epsilon^{o\sqrt{(-1)}})-\phi(\epsilon^{-o\sqrt{(-1)}})\}=\phi(E)\int_0^\infty\frac{x\,dx\sin ox}{1+x^2}$$
$$=\tfrac{1}{2}\pi\,\phi(E)\,\epsilon^{-o}=\tfrac{1}{2}\pi\,\phi(\epsilon^{-1})\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.(k).$$

Make $\phi(\epsilon^{o\sqrt{(-1)}})+\phi(\epsilon^{-o\sqrt{(-1)}})=\dfrac{\epsilon^{ao\sqrt{(-1)}}+\epsilon^{-ao\sqrt{(-1)}}}{\epsilon^{bo\sqrt{(-1)}}+\epsilon^{-bo\sqrt{(-1)}}}=\dfrac{\cos ax}{\cos bx}$

$$\phi(\epsilon^{o\sqrt{(-1)}})-\phi(\epsilon^{-o\sqrt{(-1)}})=\frac{\epsilon^{ao\sqrt{(-1)}}-\epsilon^{-ao\sqrt{(-1)}}}{\epsilon^{bo\sqrt{(-1)}}+\epsilon^{-bo\sqrt{(-1)}}}=\sqrt{(-1)}\,\frac{\sin ax}{\cos bx}.$$

Here a must be less than b, or the second members would reduce to a different form. These functional equations solved give

$$\phi(\epsilon^{o\sqrt{(-1)}})=\tfrac{1}{2}\,\frac{\epsilon^{ao\sqrt{(-1)}}+\epsilon^{-ao\sqrt{(-1)}}}{\epsilon^{bo\sqrt{(-1)}}+\epsilon^{-bo\sqrt{(-1)}}}$$

for the first, and

$$\phi(\epsilon^{o\sqrt{(-1)}})=\tfrac{1}{2}\,\frac{\epsilon^{ao\sqrt{(-1)}}-\epsilon^{-ao\sqrt{(-1)}}}{\epsilon^{bo\sqrt{(-1)}}+\epsilon^{-bo\sqrt{(-1)}}}$$

for the second. With these values of $\phi(\epsilon^{o\sqrt{(-1)}})$, (k) give immediately

$$\int_0^\infty\frac{dx}{1+x^2}\,\frac{\cos ax}{\cos bx}=\tfrac{1}{2}\pi\,\frac{\epsilon^a+\epsilon^{-a}}{\epsilon^b+\epsilon^{-b}},\quad\int_0^\infty\frac{x\,dx}{1+x^2}\,\frac{\sin ax}{\cos bx}=-\tfrac{1}{2}\pi\,\frac{\epsilon^a-\epsilon^{-a}}{\epsilon^b+\epsilon^{-b}}.$$

In (i) change ϵ^o into x, and it gives

$$\phi(x)=\phi(E)\,x^o\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.(l),$$

which is Hamilton's theorem.

Change in (l) $\phi(x)$ into $\phi(\epsilon^x)$, and then x into $\log x$; there results

$$\phi(x)=\phi(\epsilon^E)\,(lx)^o\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,(m).$$

By a repetition of the same steps, we might add more theorems; and let it be remembered that we can always replace o by r, if it appear desirable to do so.

Let $\phi(a,x)=\Sigma(A_nx^n)$; then $\left(x\dfrac{d}{dx}\right)^r\phi(a,x)=\Sigma(n^rA_nx^n)$, and consequently $\psi\left(x\dfrac{d}{dx}\right)\phi(a,x)=\Sigma\{\psi(n)A_nx^n\}$. Make

Also $\qquad \varepsilon^{D} o^{n} = 1^{n}, \quad \varepsilon^{2D} o^{n} = 2^{n}, \quad$ &c.

Therefore $\qquad \varepsilon^{xD} o^{n} = x^{n} \varepsilon^{D} o^{n}, \quad \varepsilon^{2xD} o^{n} = x^{n} \varepsilon^{2D} o^{n}, \quad$ &c.

And hence also $\qquad f(\varepsilon^{xD})\, o^{n} = x^{n} f(\varepsilon^{D})\, o^{n}.$

Or $\qquad f\{(1+\Delta)^{x}\}\, o^{n} = x^{n} f(1+\Delta)\, o^{n}.$

Or $\qquad f(xD)\, o^{n} = x^{n} f(D)\, o^{n}.$

If we expand (a) by the powers of lx, and if we change x in (l) into ε^{lx}; and expand that in like manner, and then compare like terms; we shall find

$$D^{n}\, \phi(\varepsilon^{t}) = \phi(1+\Delta)\, o^{n}.$$

We have treated only of the general form $\phi(x)$; but it is in certain particular functions, that the symbols of operation D and Δ give those simple expressions of them, which afford such easy and elegant means of integration. And here too we can sometimes employ other and more complex symbols than D and Δ with great effect.

Gunthwaite Hall, near Barnsley, Dec. 11, 1846.

ON PRINCIPAL AXES OF A BODY, THEIR MOMENTS OF INERTIA, AND DISTRIBUTION IN SPACE.

By Richard Townsend.

(*Continued from* p. 42.)

27. Any or all of the above constructions for principal axes (17, 18, 26) verify the anticipations of (2), shewing immediately, that an axis taken at random in a body may not be a principal axis at all; those axes alone being principal which are normals to surfaces of the second order confocal with the ellipsoid of gyration; and their principal points, or the points for which they are principal, being the points on these surfaces at which they are normals, and their corresponding principal planes being of course the tangent planes at those points to the same surfaces.

In distinguishing between axes in a given body, we have therefore to determine respecting every given axis, 1st, whether it be principal or not; and, 2nd, if it be, where will be its principal point, or points, if it have more than one.

Towards this we have the well-known theorem:

The normal and tangent plane at every point of any surface of the second order will meet each of its three principal planes in a point and line, which will always be pole and polar to each other with respect to the focal conic in that plane.

smaller systems according to different arbitrary laws of division, possesses many curious and interesting properties.

The fourth developable (29) remains still to be noticed: if, therefore, we take any line of curvature on any one of the whole system of surfaces confocal with the ellipsoid of gyration, the system of normal planes to that curve will generate by their successive intersections a developable surface, all whose edges will be principal axes.

31. *Every* ruled surface, whether gauche or developable, which is generated by a system of principal axes, is connected with the corresponding developable surface envelope of the corresponding system of principal planes by the following relation; from which if either be given, the other may be readily determined.

Their curves of intersection with each principal plane of the ellipsoid of gyration are always polars reciprocal to each other with respect to the focal conic in that plane.

For, the points in which the different principal axes pierce that plane are the poles with respect to the focal conic therein of the lines in which the corresponding principal planes intersect the same; the curve, therefore, envelope

these three conics, as above, will be all parabolas if the generated surface pass through the centre of gravity. And if (as not unfrequently happens in consequence of every axis which lies in either of the three planes being (28) principal) it touch any of the central principal planes, that is if it intersect it in two right lines, then apparently will the conic in which the corresponding developable intersects that plane be infinitely flat, the right line viz. which passes through the poles of the two lines with respect to the focal conic: in this case, however, since one of the two lines must be one of the generating axes, the whole right line is due to *its* principal plane, and the conic is therefore properly but a point, the pole of the other line.

If the surface of the second order generated by the system of principal axes touch the three central principal planes, then will the three conics in which the developable envelope of the corresponding system of principal planes intersects those planes, all dwindle into points; that developable therefore in this case will be an infinitely slender cylinder. Hence we see that if a surface of the second order generated by a system of principal axes touch any two of the central principal planes, it must also touch the third, and that if a system of planes pass all through the same right line, that is, if they all touch an infinitely slender cylinder of the second order, then will the surface generated by the corresponding system of principal axes touch the three central principal planes; that surface will in fact be a paraboloid of the second order. But of this more hereafter.

In general, if the developable surface envelope of any system of planes be an infinitely flat cone or cylinder of any order, and therefore intersect the central principal planes in finite portions of a right line bounded by points, then will the surface generated by the corresponding system of principal axes touch those three planes, for it intersects them in right lines, the poles of the above points, and every plane which passes through a right line on a gauche surface is a tangent plane to that surface at some point or other.

32. As an example illustrative of the preceding article, Let a system of planes touch the ellipsoid of gyration or any one of its confocal surfaces along any plane section; their developable envelope will be of course a cone of the second order, intersecting the three central principal planes in conics, and the corresponding system of principal axes, that is the system of normals to the surface along the plane curve,

paraboloids.

conic.

The complete curve of the fourth order in which the surface generated by the normals intersects each central principal plane, consists therefore in this case of four real right lines, in pairs parallel to each other and similarly situated in opposite directions from the centre of gravity, one pair of these parallel lines being generating axes, and the other two forming the polar reciprocal with respect to the focal conic of the particular curve, in which the developable envelope of the corresponding system of principal planes intersects that central plane; a result confirmatory of the concluding remarks in (31).

That such should be the nature of the intersecting curves we might easily have seen *à priori*, for in the particular case in question the gauche surface of the fourth order, generated by the system of principal axes, breaks up into two paraboloids of the second order, equal, similar, and similarly placed, but in opposite directions from the centre of gravity, one corresponding to one of the two parallel generatrices of the hyperboloid of one sheet, and the other to the second; they both touch the three central principal planes, and therefore intersect them each in two right lines, the analogous lines for each being of course parallel to each other.

two or more different and distinct groups of smaller systems according to different and arbitrary laws of division.

In every case of the division of a system of principal axes restricted by but one condition, the surfaces formed by the smaller systems, as containing each but a single parameter variable from one to the other, will admit of an envelope, this will obviously be the surface generated by their successive characteristics or the curves in which they intersect, two and two consecutively; and to find the equation of that surface, we have but to proceed in the usual manner, setting out from the equation containing but one parameter which expresses the system of surfaces enveloped.

Now, though there exists an infinite number of ways in which the division of a system of surfaces may be performed, and therefore an infinite number of groups of surfaces enveloped, still for all of them the envelope will be the same, but the circumstances of its determination will be considerably different in the different cases; these will readily appear from the following considerations.

Whenever we have a system of right lines which are restrained by any two independent conditions, or, which is the same thing, when their parameters are connected by two independent equations, that system will of course envelope

the consecutive surface, there will pass a generatrix of each of these two surfaces, that is, a consecutive pair of the original system of axes will there intersect each other. Again, of these two generatrices at each point of this curve, one will always meet the consecutive curve which lies on its own surface, and through the point of meeting there will pass also a generatrix of the third consecutive surface, that is, a third consecutive axis of the given system will there meet the second. This will again meet the third consecutive curve, and through the point of meeting there will pass a fourth consecutive axis of the given system; and so on, at curve after curve, the same thing will take place successively, until the whole series of curves will be all exhausted. Hence, passing through a point on every curve of the whole system, we shall have a developable surface formed of a system of the original axes, and hence therefore, in the transition from point to point of any one individual curve, we shall have that whole system of axes completely exhausted, and divided into a series of developable surfaces.

The suggested division of every such system of principal axes into a system of developable surfaces, as a preparatory step towards endeavouring to find the surface envelope

up into two, three, or more different surfaces; but it would be impossible that the whole system infinite in number should be all circumscribed to the two sheets simultaneously. Hence always the arêtes de rebroussement of the new system lie all on the first sheet, upon which, for the same reason as above, they are, conversely, the system of curves conjugate to the lines of contact of the former system.

Hence we see that, for every system of principal axes restricted by a single condition, there exist always two and but two different and distinct systems of developable surfaces, into either of which that whole system may be always divided; and also that the surface envelope of every such system of axes consists generally of two different and distinct sheets, separated from, and rarely if ever running into each other, of which sheets each will be enveloped by one of the two component systems of developable surfaces into which that system of axes may be divided, and will be the locus of the arêtes de rebroussement of the other system, and upon both of which the two opposite systems of generating curves, the lines of contact and the lines of regression, will be always conjugate to each other. These properties bear an obvious and close analogy to those of the whole system of normals to every algebraic surface, for

[redacted]

$$\Lambda(-x, a) = -\Lambda(x, \pi - a),$$

which subsists by virtue of the convention (already proposed) that *log x* is always to mean $\frac{1}{2}\log(x^2)$.

$$\text{Assuming } x = \frac{\sin\omega}{\sin(\omega + a)}, \text{ or } \tan\omega = \frac{x\sin a}{1 - x\cos a},$$

$$\text{we get } \sqrt{X} = \frac{\sin a}{\sin(\omega + a)};$$

$$\text{whence } \Lambda(x, a) = \int \log\frac{\sin(\omega + a)}{\sin\omega} \,.\, d\log\sin(\omega + a);$$

which we shall hereafter denote by $\chi(\omega, a)$; so that $\Lambda(x, a)$ and $\chi(\omega, a)$ are identical forms. This substitution is chiefly of use in enabling us to understand the nature of other transformations at which we shall arrive. For the present, when ω is named, it is supposed to bear this relation to x.

2. To find the *complete function* $\Lambda(1, a)$, which $= -\lambda(1, a)$.

$$\text{Since } \lambda(x, a) = \tfrac{1}{2}\int_0 \log X\frac{dx}{x}, \quad \frac{d\lambda}{da} = \int_0 \frac{\sin a}{X}\,dx = \omega.$$

$$\text{Make } x = 1, \therefore \tan\omega = \frac{\sin a}{1 - \cos a} = \cot\tfrac{1}{2}a, \text{ or } \omega = \tfrac{1}{2}(\pi - a).$$

Integrate $\dfrac{d\lambda}{da} = \frac{1}{2}(\pi - a)$; $\therefore \lambda(1, a) = c + \frac{1}{2}\pi a - \frac{1}{4}a^2$.

To find c, make $a = 0$; $\lambda(x, 0) = \int_0 \log(1 - x)\dfrac{dx}{x} = L(1 - x)$,

whence $\lambda(1, 0) = L0$, or $c = -\frac{1}{6}\pi^2$,

$$\left.\begin{aligned} &\therefore\ \Lambda(1, a) = -\lambda(1, a) = \tfrac{1}{6}\pi^2 - \tfrac{1}{2}\pi a + \tfrac{1}{4}a^2. \\ &\text{Hence also }\ \Lambda(1, \pi - a) = \tfrac{1}{4}a^2 - \tfrac{1}{12}\pi^2. \end{aligned}\right\}\ldots.(1).$$

3. To find Λ at special values of a.

At the extreme values, ($a = 0$ and $a = \pi$), X is an algebraic square, $(x \pm 1)^2$. Hence

$$\left.\begin{aligned} &\Lambda(x, 0) = Lx - L0 \\ &\Lambda(x, \pi) = L(-x) - L0 = lxl(1 + x) - L(1 + x) \end{aligned}\right\}\ldots(2).$$

In the following process, we for a moment suppose a to increase from 0 to any magnitude; and, to shew both variables, write $f(x, a)$ instead of X. Then, by a well-known formula of Trigonometry,

$$f(x^n, na) = f(x, a).f\left(x, a + \frac{2\pi}{n}\right).f\left(x, a + \frac{4\pi}{n}\right)\ldots f\left(x, a + \frac{2n-2}{n}\pi\right).$$

Differentiate logarithmically: multiply by $(2n)^{-1}.\log(x^n) = 2^{-1}\log x$, and integrate: then

$$\frac{1}{n}\Lambda(x^n, na) = \Lambda(x, a) + \Lambda\left(x,\ a + \frac{2\pi}{n}\right)$$

$$+\Lambda\left(x, a + \frac{4\pi}{n}\right) + \ldots. + \Lambda\left(x, a + \frac{2n-2}{n}\pi\right)\ldots.(3).$$

In particular, if $n = 2$, $\cos(a + \pi) = \cos(\pi - a)$,

$$\left.\begin{aligned} \therefore\ \ \tfrac{1}{2}\Lambda(x^2, 2a) &= \Lambda(x, a) + \Lambda(x, \pi - a) \\ &= \Lambda(x, a) + \Lambda(-x, a) \end{aligned}\right\}\ldots.(4),$$

which has a certain analogy to $Lx + L(-x) = \frac{1}{2}L(x^2) + \frac{3}{2}L0$.

When $a = \dfrac{\pi}{n}$, we have

$$\left.\begin{aligned} &\frac{1}{n}\Lambda(x^n, \pi)\text{ or }L(-x^n) - L0 \\ &= \Lambda\left(x, \frac{\pi}{n}\right) + \Lambda\left(x, \frac{3\pi}{n}\right) + \Lambda\left(x, \frac{5\pi}{n}\right) + \ldots + \Lambda\left(x, \frac{2n-1}{n}\pi\right) \end{aligned}\right\}.$$

Introvert the terms; and to make every a fall between 0 and π, observe that $\cos\dfrac{2n-r}{n}\pi = \cos\dfrac{r\pi}{n}$. Then we find that

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$$\left.\begin{array}{ll}\text{If } n=2, & \Lambda(x, \tfrac{1}{2}\pi) = \tfrac{1}{4}\Lambda(x^2, \pi): \\ \text{If } n=3, & \Lambda(x, \tfrac{1}{3}\pi) = \tfrac{1}{6}\Lambda(x^3, \pi) - \tfrac{1}{2}\Lambda(x, \pi)\end{array}\right\} \ldots (6).$$

If in equation (3) we make $na = 2\pi$, and n is odd, we similarly have

$$\Lambda\left(x, \frac{2\pi}{n}\right) + \Lambda\left(x, \frac{4\pi}{n}\right) + \ldots\ldots + \Lambda\left(x, \frac{n-1}{n}\pi\right)$$

$$= \frac{1}{2n}\Lambda(x^n, 0) - \tfrac{1}{2}\Lambda(x, 0) \ldots\ldots (7):$$

If $n = 3$ in the last,

$$\Lambda(x, \tfrac{2}{3}\pi) = \tfrac{1}{6}\Lambda(x^3, 0) - \tfrac{1}{2}\Lambda(x, 0) \ldots\ldots\ldots (8).$$

Thus we know $\Lambda(x, a)$ in finite functions of x, by means of L, when a has any of the values 0, π, $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, $\frac{2}{3}\pi$. It will afterwards appear, that the assertion may be extended to the case of $a =$ any of these, *divided by* 2^n, when n is an arbitrary integer.

4. To find Λ, when x (at the upper limit) is a given function of a.

Generally, if $u = Fx$, $V = \psi(x, a)$, and $U = \int u\,dV$,

$$\frac{dU}{da} = \int u\,\frac{d^2V}{dx\,da}\,dx = \int u\,\frac{dV'}{dx}\,dx, \text{ if } V' = \frac{dV}{da};$$

Integrate by parts, and the last is $(uV' - \int V'du)$; and the *total* differential

$$d(U) = \frac{dU}{dx}\,dx + \frac{dU}{da}\,da = u\,\frac{dV}{dx}\,dx + \left(u\,\frac{dV}{da} - \int V'du\right)da$$

$$= u.d(V) - (\int V'du)\,da.$$

In the present case, let $u = \frac{1}{2}\log x$, $V = \log X$, and observe that $V' = 2xX^{-1}\sin a$, which vanishes with x; and $\int V'du = \omega$, which also vanishes with x, as $\dfrac{dU}{da}$ or (here) $\dfrac{d\Lambda}{da}$ ought to do

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$$2\Lambda \cos a - \Lambda(2\cos a) = \tfrac{1}{2} L \cos^2 a - \tfrac{1}{2} L0,$$

which is verified by (10) and (11).

Make $x = 1$, and we find

$$\Lambda(2\cos a - 1) = \tfrac{1}{2} L(2\cos a - 1) + \tfrac{1}{6}\pi^2 - \tfrac{1}{6}3\pi a + \tfrac{1}{4}5a^2 \dots (14^*).$$

We may combine the two last integrations, by supposing $x^{-1} + y = 2\cos a$, which gives

$$\Lambda x - \Lambda y + \tfrac{1}{2}a^2 = \tfrac{1}{2}\log^2 x - \tfrac{1}{2} L\frac{y}{x} \;\dots\dots(15).$$

Again, in this write y^{-1} for y, and eliminate Λy^{-1} by means of (13), and $L(x^{-1}y^{-1})$ by means of the known properties of L; then

$$\left\{\begin{array}{l} x^{-1} + y^{-1} = 2\cos a\,; \\ \Lambda x + \Lambda y - \pi(\tfrac{1}{2}\pi - a) = \tfrac{1}{2} L(xy) + \tfrac{1}{4}\log^2\left(\dfrac{x}{y}\right) \end{array}\right\} \dots (16).$$

But in this, the arbitrary constant is liable to change by reason of discontinuity, if x or y passes through zero.

7. The four suppositions here made have something in common. In (13), (14), and (16), we find

$$\frac{dx}{X} = \frac{dy}{Y}\,; \text{ and in (15), } \frac{dx}{X} = -\frac{dy}{Y}.$$

Let us in all suppose $x = \dfrac{\sin\omega}{\sin(\omega + \alpha)}$;

then if $y = \dfrac{\sin\theta}{\sin(\theta - \alpha)}$, $xy = 1$, when $\theta = \omega + \alpha$

......, but $x + y = 2\cos\alpha$, when $\theta = \omega + 2\alpha$.

8. By equation (13) we can obtain $\Lambda(\sec\alpha)$ and $\Lambda(\frac{1}{2}\sec\alpha)$ from $\Lambda(\cos\alpha)$ and $\Lambda(2\cos\alpha)$. Observing that

$$L\cos^2\alpha + L\sec^2\alpha = 2\log^2\cos\alpha,$$

we have
$$\left.\begin{aligned}\Lambda\sec\alpha &= \tfrac{1}{4}L\sec^2\alpha + \tfrac{1}{2}\pi(\tfrac{1}{3}\pi - \alpha)\\ \Lambda(\tfrac{1}{2}\sec\alpha) &= \tfrac{1}{2}\log^2(\tfrac{1}{2}\sec\alpha) + \tfrac{1}{12}\pi^2 - \tfrac{1}{2}\alpha^2\end{aligned}\right\}\ldots.(17).$$

9. Farther, since we fulfil the relation $x^{-1} + y = 2\cos\alpha$, by supposing

$$x = \frac{\sin\omega}{\sin(\omega+\alpha)},\quad y = \frac{\sin(\omega - \alpha)}{\sin\omega},$$

it is evident that if Λx is known, we can by the repeated use of (15) find $\Lambda\dfrac{\sin\{\omega - (n+1)\alpha\}}{\sin(\omega - n\alpha)}$. Or conversely, if Λy is known, we can deduce $\Lambda\dfrac{\sin(\omega + n\alpha)}{\sin\{\omega + (n+1)\alpha\}}$.

For example: *first*, let m_n stand for $\dfrac{\cos n\alpha}{\cos(n-1)\alpha}$; then $m_n^{-1} + m_{n+1} = 2\cos\alpha$;

$$\therefore\ 2\Lambda m_{n+1} - 2\Lambda m_n = \alpha^2 + L(m_n^{-1}m_{n+1}) - \log^2 m_n.$$

For n write $1, 2, 3 \ldots (n-1)$, and add the results, taking Λm_1 from (11);

$$\left.\begin{aligned}\therefore\ 2\Lambda\frac{\cos n\alpha}{\cos(n-1)\alpha} &= \tfrac{1}{3}\pi^2 - \pi\alpha + n\alpha^2\\ &+ \tfrac{1}{2}L\cos^2\alpha + L(m_1^{-1}m_2) + L(m_2^{-1}m_3) + \ldots + L(m^{-1}_{n-1}m_n)\\ &- \log^2 m_1 - \log_2 m_2 - \&c.\ldots\ldots - \log^2 m_{n-1}:\end{aligned}\right\}$$

where for $\cos^2\alpha$ we may write $(m_0^{-1}m_1)$.

Similarly, if $m_n = \dfrac{\sin n\alpha}{\sin(n+1)\alpha}$, $m_n^{-1} + m_{n-1} = 2\cos\alpha$; and Λm_1 is known by (17); so that

$$2\Lambda\frac{\sin n\alpha}{\sin(n+1)\alpha} = \tfrac{1}{6}\pi^2 - n\alpha^2 - L\frac{m_1}{m_2} - \ldots\ldots - L\frac{m_{n-1}}{m_n}$$
$$+ \log^2 m_1 + \log^2 m_2 + \ldots\ldots + \log^2 m_n.$$

the second

$$= lml(m - x) + L\frac{x}{m}; \text{ the third} = lml(m - y) + L\frac{y}{m}.$$

Observe that $lml(m - x) + lml(m - y) = lmlM = \text{const.}$;

$$\therefore\ 2\Lambda x + 2\Lambda y - \text{const.} = L(xy) + L\frac{x}{m} + L\frac{y}{m}.$$

Let $x = e$, when $y = 0$; then $e = 2\cos\alpha - m^{-1}$; so that Λe and Λm will be known from one another by (15). Also

$$\left(1 - \frac{x}{m}\right)\left(1 - \frac{y}{m}\right) = 1 - \frac{e}{m} = 1 - 2e\cos\alpha + e^2 = E.$$

Finally

$$2(\Lambda x + \Lambda y - \Lambda e) = L(xy) + L\frac{x}{m} + L\frac{y}{m} - L\frac{e}{m} - 2L0 \ldots (18).$$

By slightly varying the integration, we get

$$2(\lambda x + \lambda y - \lambda e)$$
$$= L(1 - xy) + L\left(1 - \frac{x}{m}\right) + L\left(1 - \frac{y}{m}\right) - L\left(1 - \frac{e}{m}\right) \ldots (18^*),$$

which may be convenient when m, or one of the variables, is negative.

By giving special values to m, such as make Λm and Λe known functions, the equations become available to us in many ways. But in order to understand our result, it will be well to transform it by means of ω and the function χ of Art. 1.

$\Lambda(x, \frac{1}{2}\pi)$.

14. By a repeated use of the equation of bisection, it is evident that $\Lambda(x, \alpha)$ is reducible to $\Lambda(x_n, 2^{-n}\alpha)$, which, when $n = \infty$, is $\Lambda(x_n, 0)$ a known function. It may be worth while to enter into a few details concerning this.

Let α_n represent $2^{-n}\alpha$, and from x suppose $x_1, x_2 \ldots$ to be derived by the law

$$2x_1 \cos \alpha_1 = 1 + x; \; 2x_2 \cos \alpha_2 = 1 + x_1; \; \text{\&c. \&c.} \ldots$$

It is easy to compute these by the intervention of ω. For we had

$$\omega + \alpha = \theta + \beta \text{ or } = \omega_1 + \alpha_1 = \omega_2 + \alpha_2 = \omega_3 + \alpha_3 = \text{\&c.},$$

$$\text{whence } \omega_n = \omega + \alpha - \alpha_n.$$

Thus $x_n = \dfrac{\sin(\omega + \alpha - 2^{-n}\alpha)}{\sin(\omega + \alpha)}$, which, when $n = \infty$, converges to 1, and nearly $= 1 - 2^{-n}\alpha \cot(\omega + \alpha)$. (We must entirely except the case of $\omega + \alpha = 0$ or $= \pi$, which gives $x = \infty$.) Hence $2^n . \Lambda(x_n \alpha_n) = 2^n . \Lambda(x_n 0) = 2^n . \{Lx_n + \frac{1}{6}\pi^2\} = 2^n . (x_n - 1) + 2^n . \frac{1}{6}\pi^2 = -\alpha \cot(\omega + \alpha) + 2^n . \frac{1}{6}\pi^2$. Apply equation (21) n times: multiply the results by $2^0, 2^1, 2^2 \ldots 2^{n-1}$, and add all together. Substitute for $2^n . \Lambda(x_n \alpha_n)$ as above, and be careful to note that $2^{-2} + 2^{-3} + \ldots + 2^{-\infty} = \frac{1}{2}$,

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* It is easy to combine equations (20), (21) with (13).

places without great labour; and some of the following methods may become preferable.

§ V.—*To take advantage of a lying within certain limits.*

17. If a is extremely small, and x is $< \frac{1}{2}$; or if, x being near to 1, the product $2 \sin \frac{1}{2}a \cdot \left(\frac{x}{1-x}\right)$ is still very small.

Put $b = 2 \sin \frac{1}{2}a$, $z = \dfrac{x}{1-x}$, or $x = \dfrac{z}{1+z}$; $1 - x = \dfrac{1}{1+z}$;

$$X = (1-x)^2 + b^2 x = (1-x)^2 . \{1 + b^2 z . (1+z)\}$$

$$d \log x = d \{\log z - \log (1+z)\} = \frac{dz}{z(1+z)};$$

$$\therefore \lambda(x, a) = \tfrac{1}{2}\int_0 \log X \, d \log x = \int_0 \log (1-x) \, d \log x$$
$$+ \tfrac{1}{2}\int_0 \log \{1 + b^2 z . (1+z)\} \frac{dz}{z(1+z)}$$
$$= L(1-x) + \tfrac{1}{2}P \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23),$$

if $P = \int_0 \{b^2 - \frac{1}{2}b^4 z . (1+z) + \frac{1}{3}b^6 z^2 . (1+z)^2 - \&c. \ldots\} \, dz$

$$= b^2 z - \tfrac{1}{2}b^4 (\tfrac{1}{2}z^2 + \tfrac{1}{3}z^3) + \tfrac{1}{3}b^6 (\tfrac{1}{3}z^3 + 2\,\tfrac{1}{4}z^4 + \tfrac{1}{5}z^5) - \&c. \ldots (23^*),$$

which converges rapidly, since bz is very small.

18. If, on the contrary, a is very near to π (which is always the more favourable case, x being supposed positive), let $x = \tan^2 \frac{1}{2}\omega$; then $\lambda(x, a) = L(1+x) - 2\Omega$,

$$\text{if } \Omega = -\tfrac{1}{2}\int_0 \log(1 - \cos^2 \tfrac{1}{2}a \sin^2 \omega)\frac{d\omega}{\sin\omega}.$$

If we develop the logarithm, we readily see that Ω may take the form

$$A_0 - 2A_1 \cos\omega + 2A_3 \frac{\cos 3\omega}{3} - 2A_5 \frac{\cos 5\omega}{5} + \&c.$$

To find A_0, let $\omega = \frac{1}{2}\pi$, $\Omega = A_0$, $x = 1$, $\therefore \lambda(1, a) = L2 - 2A_0$; whence $2A_0 = \dfrac{\pi^2}{12} + \Lambda(1, a) = \left(\dfrac{\pi - a}{2}\right)^2$.—Let $\pi - a = 4\beta$,

$$\therefore A_0 = 2\beta^2.$$

Next $\dfrac{d\Omega}{d\omega} = 2A_1 \sin\omega - 2A_3 \sin 3\omega + 2A_5 \sin 5\omega - \&c.$,

$$\text{also } \frac{d\Omega}{d\omega} = -\tfrac{1}{2}\log(1 - \sin^2 2\beta\,.\,\sin^2\omega)\frac{1}{\sin\omega}.$$

Put $b = \tan\beta$, $\sin 2\beta = \dfrac{2b}{1+b^2}$; and the value of $\sin\omega\,.\,\dfrac{d\Omega}{d\omega}$

$$\text{is } \log(1+b^2) - \tfrac{1}{2}\log(1 + 2b^2\cos 2\omega + b^4),$$

or $\log(1+b^2) - b^2\cos 2\omega + \frac{1}{2}b^4\cos 4\omega - \frac{1}{3}b^6\cos 6\omega + \&c.$, which is to be made equal to

$$2\sin\omega\,\{A_1\sin\omega - A_3\sin 3\omega + A_5\sin\omega - \&c.\},$$

or $A_1(1-\cos 2\omega) - A_3(\cos 2\omega - \cos 4\omega) + A_5(\cos 4\omega - \cos 6\omega) - \&c.$
Hence we get $\quad A_1 = \log(1+b^2)$,

$$A_1 + A_3 = \frac{b^2}{1}; \text{ and generally } A_{2n-1} + A_{2n+1} = \frac{b^{2n}}{n}.$$

In the First Part of these investigations we have used $\phi_n x$ to denote $\int_0 \tan^{n-1}x\,dx$; which yields $\phi_1 x = x$, $\phi_2 x = \frac{1}{2}\log(1+\tan^2 x)$ or $\log\sec x$; and $\phi_n x + \phi_{n+2}x = \dfrac{\tan^n x}{n}$.

Thus $A_1 = 2\phi_2\beta$; $A_3 = 2\phi_4\beta$; $A_5 = 2\phi_6\beta$; &c. . . .

$$\left.\begin{aligned}\text{and } \lambda(x, a) = L(1+x) - 4\beta^2 + 8\phi_2\beta\,.\,\frac{\cos\omega}{1} - 8\phi_4\beta\,.\,\frac{\cos 3\omega}{3}\\ + 8\phi_6\beta\,.\,\frac{\cos 5\omega}{5} - 8\phi_8\beta\,.\,\frac{\cos 7\omega}{7}\\ + \&c.\ldots \&c.\ldots\end{aligned}\right\}\ldots(24),$$

which converges best when β is least, or a nearest to π.

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Farther, put $c = \tan\gamma$, $\cos\alpha = \sin 2\gamma = \dfrac{2c}{1+c^2}$;

$$\cos\omega.\frac{d\Omega}{d\omega} = \frac{1}{2}\log\left(1 - \frac{2c\cos\omega}{1+c^2}\right)$$

$$= -\frac{1}{2}\log(1+c^2) - c\cos\omega - \frac{1}{2}c^2\cos 2\omega - \frac{1}{3}c^3\cos 3\omega - \&c. \ldots$$

But $-\cos\omega\dfrac{d\Omega}{d\omega} = \cos\omega\left\{C_0 + 2C_1\cos\omega + 2C_2\cos 2\omega + \&c. \ldots\right\}$

$$= C_1 + (C_0+C_2)\cos\omega + (C_1+C_3)\cos 2\omega + (C_2+C_4)\cos 3\omega + \&c. \ldots$$

$$\therefore\ C_1 = \frac{1}{2}\log(1+c^2) = \phi_2\gamma;\quad C_2 = c - C_0 = \tan\gamma - \gamma = \phi_3\gamma;$$

$$C_3 = \frac{1}{2}c^2 - C_1 = \phi_4\gamma;\qquad C_4 = \frac{1}{3}c^3 - C_2 = \phi_5\gamma;\ \&c. \ldots$$

$$\left.\begin{aligned}\text{Whence}\ \ \lambda(x, \alpha) = \tfrac{1}{2}L(1+x^2) - (\tfrac{1}{2}\pi\gamma + \gamma^2) + \gamma\omega \\ + 2\phi_2\gamma.\frac{\sin\omega}{1} + 2\phi_3\gamma.\frac{\sin 2\omega}{2} + 2\phi_4\gamma.\frac{\sin 3\omega}{3} + \&c.\end{aligned}\right\}\ldots(25),$$

where $\gamma = \frac{1}{2}\pi - \frac{1}{2}\alpha$, $x = \tan(\frac{1}{4}\pi - \frac{1}{2}\omega)$.
The convergence is rapid when α is very near to $\frac{1}{2}\pi$.

In equation (24), put $\omega = 0$, $\Omega = 0$;

$$\therefore\ \tfrac{1}{4}\beta^2 = \phi_2\beta - \tfrac{1}{3}\phi_4\beta + \tfrac{1}{5}\phi_6\beta - \&c.$$

In the value of $\dfrac{d\Omega}{d\omega}$ corresponding, make $\omega = \frac{1}{2}\pi$;

$$\therefore -\tfrac{1}{2}\log\cos 2\beta = \phi_2\beta + \phi_4\beta + \phi_6\beta + \&c. \ldots$$

In equation (25), if we change γ into $-\gamma$, $\phi_{2n}\gamma$ remains unchanged, and $\phi_{2n-1}\gamma$ changes sign. By adding the two results thus obtained, we might easily reproduce equation (24). Put $\omega = \pi$ in the value of $\dfrac{d\Omega}{d\theta}$ corresponding to (25);

$$\therefore \tfrac{1}{2}\log(1 + \sin 2\gamma) = \phi_1\gamma - 2\phi_2\gamma + 2\phi_3\gamma - 2\phi_4\gamma + \&c.,$$

$$\text{so } \tfrac{1}{2}\log(1 - \sin 2\gamma) = -\phi_1\gamma - 2\phi_2\gamma - 2\phi_3\gamma - 2\phi_4\gamma - \&c.;$$

which gives not only

$$-\tfrac{1}{4}\log\cos 2\gamma = \phi_2\gamma + \phi_4\gamma + \phi_6\gamma + \&c.,$$

but also $\tfrac{1}{2}\log\tan(\tfrac{1}{4}\pi + \gamma) = \phi_1\gamma + 2\phi_3\gamma + 2\phi_5\gamma + \&c.\ldots.$

These are mere properties of the functions $\phi_1, \phi_2, \phi_3 \ldots$ and can in several ways be verified.

The series (24), (25) cannot be practically used with advantage, unless we have tables of $\phi_n a$; but these might be computed with so much ease, within the limits $a = 0$, $a = 45°$, that this is apparently the best method of adding completeness to this branch of the calculus. The following section will shew that the use of ϕ_n is not confined to the particular cases contemplated in equations (24), (25).

§ VI.—*To find* Λ, *when* x *is near to* 1.

20. We shall suppose a to be $< 90°$, and deal with

$\Lambda(x, \pi - a)$ and $\Lambda(x, a)$ separately.

Put $\cos a = \dfrac{1 - m^2}{1 + m^2}$, or $m = \tan\tfrac{1}{2}a$; $X' = 1 + 2x\cos a + x^2$;

$$\therefore (1 + m^2)X' = (1 + x)^2 + m^2(1 - x)^2. \quad \text{Let } y = \frac{1 - x}{1 + x}.$$

Then $\Lambda(x, \pi - a) = \tfrac{1}{2}\log x \log X' - L(1 + x) + R$,

$$\text{if } R = \int \log\frac{1 + m^2y^2}{1 + m^2} \cdot \frac{dy}{1 - y^2}.$$

$$\text{Assume } -\frac{dR}{dy} = M_0 - M_2y^2 + M_4y^4 - \&c.\ldots;$$

$$\therefore \log(1 + m^2) - \frac{m^2y^2}{1} + \frac{m^4y^4}{2} - \frac{m^6y^6}{3} + \&c.\ldots.$$

$$= (1 - y^2)\{M_0 - M_2y^2 + M_4y^4 - \&c.\ldots.\},$$

which gives

$M_0 = \log(1 + m^2) = 2\phi_2\tfrac{1}{2}a$; $M_2 = 2\phi_4\tfrac{1}{2}a$; $M_4 = 2\phi_6\tfrac{1}{2}a$; &c....

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ACG	BDG	CEG	DCH	ECD	FCE	GCF						
ADF	BEF	CFH	DFG	EGH	FDH	GDE						
AEH	BCa	CCb	DDd	EEf	FFh	GGk						
Aal	BHm	CDc	DEe	EFg	FGi	GHl						
Abk	Bbc	Cad	Dah	Eam	Fae	Gal	Hab	Eak	Daf	Cal	Fac	Gag
Aci	Bde	Cel	Dbg	Ebl	Fbd	Gbh	Hcd	Ebi	Dbe	Cdk	Fbm	Gbf
Adh	Bfg	Cfk	Dcf	Eck	Fcm	Gcg	Hef	Ech	Dgl	Cei	Fdl	Gce
Aeg	Bhi	Cgl	Dil	Edi	Ffl	Gdf	Hgh	Edg	Dhk	Cfh	Fek	Gdm
Afm	Bkl	Chm	Dkm	Ekh	Fgk	Gem	Hik	Elm	Dim	Cgm	Ffl	Ghl

And it is plain that $V_{25} = 0$.

If now the triads be collected in which the letters a and l occur, and those letters be erased, we obtain the following circle of 22 duads,

$$B'C\ Cc\ cF\ Fd\ dC'\ C'e\ eF'\ F'f\ fD\ Dg\ gG\ Gh$$
$$hD'\ D'i\ iG'\ G'H\ Hb\ bE'\ E'm\ mE\ Ek\ kB',$$

which proves that $(v_{23} = C_{22})$. In the same way this may be verified for any number.

Having established the two fundamental propositions,

If $V_n = 0$, $\quad V_{2n+1} = 0$, and $(v_{2n-1} = C_{2n-2})$;

If $(v_{n+1} = C_n)$, $\quad V_{2n+1} = 0$, and $(v_{2n-1} = C_{2n-2})$;

we deduce the following, of which the first is self-evident:

$$V_3 = 0;$$

Because $V_3 = 0$, $\quad V_7 = 0$, and $(v_5 = C_4)$;

" $(v_5 = C_4)$, $\quad V_9 = 0$, and $(v_7 = C_6)$;

" $(v_7 = C_6)$, $\quad V_{13} = 0$, and $(v_{11} = C_{10})$;

" $V_7 = 0$, $\quad V_{15} = 0$, and $(v_{13} = C_{12})$;

" $V_9 = 0$, $\quad V_{19} = 0$, and $(v_{17} = C_{16})$;

" $(v_{11} = C_{10})$, $\quad V_{21} = 0$, and $(v_{19} = C_{18})$;

" $(v_{13} = C_{12})$, $\quad V_{25} = 0$, and $(v_{23} = C_{22})$; &c.

Generally, $V_{6n+1} = 0 = V_{6n+3}$, for all values of n.

determinants, the former with the differential coefficients of these functions with respect to u, v,....and the latter with the differential coefficients of the same functions with respect to x, y,.....the quotient with its sign changed obtained by dividing the first of these determinants by the second is, as is well known, the value of the function ∇.

Putting for shortness

$$\frac{dx}{du} = \alpha, \quad \frac{dy}{du} = \beta \ldots \quad \frac{dx}{dv} = \alpha', \quad \frac{dy}{dv} = \beta' \ldots \text{ \&c.}$$

and
$$\frac{du}{dx} = A, \quad \frac{du}{dy} = B \ldots \quad \frac{dv}{dx} = A', \quad \frac{dv}{dy} = B' \ldots$$

∇ is the reciprocal of the determinant formed with $A, B \ldots$; $A', B' \ldots\ldots$, &c. Or it is the determinant formed with $\alpha, \beta, \ldots \alpha', \beta' \ldots$, &c.

From the first of these forms, *i.e.* considering ∇ as a function of A, B..

$$\frac{d\nabla}{dA} = -\nabla\alpha, \quad \frac{d\nabla}{dB} = -\nabla\beta \ldots \quad \frac{d\nabla}{dA'} = -\nabla\alpha', \quad \frac{d\nabla}{dB'} = -\nabla\beta',$$

where the quantities $\alpha, \beta, ..\alpha', \beta', ..$and $A, B, ..A', B', ..$ may be interchanged provided $-\nabla$ be substituted for ∇. (Demonstrations of these formulæ or of some equivalent to them will be found in Jacobi's memoir "De determinantibus functionalibus," Crelle, t. XXII).

Hence

$$\frac{1}{\nabla} d\nabla + \alpha dA + \beta dB \ldots + \alpha' dA' + \beta' dB' \ldots = 0.$$

or reducing by

$$\frac{dA}{dy} = \frac{dB}{dx} \ldots \quad \frac{dA'}{dy} = \frac{dB'}{dx} \ldots \text{ \&c.}$$

this becomes

$$\left.\begin{aligned} &\frac{1}{\nabla} d\nabla + \alpha\left(\frac{dA}{dx}dx + \frac{dB}{dx}dy + \ldots\right) + \beta\left(\frac{dA}{dy}dx + \frac{dB}{dy}dy + \ldots\right) \ldots \\ &\quad + \alpha'\left(\frac{dA'}{dx}dx + \frac{dB'}{dx}dy + \ldots\right) + \beta'\left(\frac{dA'}{dy}dx + \frac{dB'}{dy}dy + \ldots\right) \ldots \end{aligned}\right\} = 0,$$

Or reducing

$$\frac{1}{\nabla} d\nabla + \left(\frac{dA}{du} + \frac{dA'}{dv} + \ldots\right) dx + \left(\frac{dB}{du} + \frac{dB'}{dv} + \ldots\right) dy + \ldots = 0;$$

whence separating the differentials and replacing $A, A', \ldots$ $B, B', \ldots$ by their values

P 2

$$U = X\frac{du}{dx} + Y\frac{du}{dy} + \dots$$

$$V = X\frac{dv}{dx} + Y\frac{dv}{dy} + \dots$$

terms of u, v,.. Then

$$= X\left(\frac{}{du}\cdot\frac{}{dx}+\frac{}{dv}\cdot\frac{}{dx}+\dots\right)+Y\left(\frac{d}{du}\cdot\frac{du}{dy}+\frac{d}{dv}\cdot\frac{dv}{dy}+\dots\right)\dots$$

$$+\left(\frac{dX}{du}\cdot\frac{du}{dx}+\frac{dX}{dv}\cdot\frac{dv}{dx}+\dots\right)+\left(\frac{dY}{du}\cdot\frac{du}{dy}+\frac{dY}{dv}\cdot\frac{dv}{dy}+\dots\right)\dots$$

i.e. $\nabla\cdot\left(\frac{dU}{du}+\frac{dV}{dv}+\dots\right)$

$$= -\left(X\frac{d\nabla}{dx}+Y\frac{d\nabla}{dy}+\dots\right)+\nabla\cdot\left(\frac{dX}{dx}+\frac{dY}{dy}+\dots\right).$$

Also, whatever be the value of M,

$$U\frac{dM\nabla}{du}+V\frac{dM\nabla}{dv}+\dots = X\cdot\frac{dM\nabla}{dx}+Y\frac{dM\nabla}{dy}+\dots$$

And from these two properties,

$$\frac{dM\nabla U}{du}+\frac{dM\nabla V}{du}+\dots = \nabla\cdot\left(\frac{dMX}{dx}+\frac{dMY}{dy}+\dots\right).$$

§ 3. Consider the system of differential equations

$$dx : dy : dz \dots\dots = X : Y : Z \dots\dots$$

(where, for greater clearness, an additional letter z has been introduced). From these we deduce the equivalent system

$$du : dv : dw \dots\dots = U : V : W \dots\dots$$

Suppose that u and v continue to represent arbitrary functions of x, y, z, but that the remaining function w,...is such as to

$$\delta M + M\left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} \dots\right) = 0,$$

where for shortness

$$\delta = \frac{d}{dt} + x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz} + \dots.$$

To reduce this we must first determine the values of λ, μ. . , and for this we have

$$\frac{d^2\Theta}{dt^2} = \delta^2\Theta + \frac{d\Theta}{dx}\cdot\frac{d^2x}{dt^2} + \frac{d\Theta}{dy}\cdot\frac{d^2y}{dt^2} + \dots = 0, \text{ \&c.}$$

i.e. $$\delta^2\Theta + P\frac{d\Theta}{dx} + Q\frac{d\Theta}{dy} + \dots + a\lambda + h\mu + g\nu \dots = 0,$$

$$\delta^2\Phi + P\frac{d\Phi}{dx} + Q\frac{d\Phi}{dy} + \dots + h\lambda + b\mu + f\nu \dots = 0,$$

$$\delta^2\Psi + P\frac{d\Psi}{dx} + Q\frac{d\Psi}{dy} + \dots + g\lambda + f\mu + c\nu \dots = 0.$$

$\vdots$

where for greater clearness an additional letter of the series Θ, Φ. . . . has been introduced, and where

$$a = \left(\frac{d\Theta}{dx}\right)^2 + \left(\frac{d\Phi}{dy}\right)^2 + \dots.$$

$$b = \left(\frac{d\Phi}{dx}\right)^2 + \left(\frac{d\Phi}{dy}\right)^2 + \dots.$$

$$\vdots$$

$$h = \left(\frac{d\Theta}{dx}\cdot\frac{d\Phi}{dx} + \frac{d\Theta}{dy}\cdot\frac{d\Phi}{dy}\right) + \dots.$$

$$\vdots$$

Hence differentiating with respect to x',

$$2\delta\frac{d\Theta}{dx} + a\frac{d\lambda}{dx'} + h\frac{d\mu}{dx'} + g\frac{d\nu}{dx'}\dots. = 0,$$

$$2\delta\frac{d\Phi}{dx} + h\frac{d\lambda}{dx'} + b\frac{d\mu}{dx'} + f\frac{d\nu}{dx'}\dots. = 0,$$

$$2\delta\frac{d\Psi}{dx} + g\frac{d\lambda}{dx'} + f\frac{d\mu}{dx'} + c\frac{d\nu}{dx'}\dots. = 0.$$

$$\vdots$$

Or representing by K the determinant formed with the quantities a, h, g. . ; h, b, f. . g, f, c. . and by A, H, G. . . . H, B, F. . . G, F, C. . . the inverse system of coefficients, we have

$$2\left(A\delta\frac{d\Theta}{dx} + H\delta\frac{d\Phi}{dx} + G\delta\frac{d\Psi}{dx}..\right) + K\frac{d\lambda}{dx'} = 0,$$

$$2\left(H\delta\frac{d\Theta}{dx} + B\delta\frac{d\Phi}{dx} + F\delta\frac{d\Psi}{dx}..\right) + K\frac{d\mu}{dx'} = 0,$$

$$2\left(G\delta\frac{d\Theta}{dx} + F\delta\frac{d\Phi}{dx} + C\delta\frac{d\Psi}{dx}..\right) + K\frac{d\nu}{dx'} = 0;$$

$$\vdots$$

whence multipling by $\frac{d\Theta}{dx}, \frac{d\Phi}{dx}, \frac{d\Psi}{dx}$ . . . . and adding

$$A\delta\left(\frac{d\Theta}{dx}\right)^2 + B\delta\left(\frac{d\Phi}{dx}\right)^2 \dots. + 2H\delta\frac{d\Theta}{dx}\cdot\frac{d\Phi}{dx}\dots. + K\frac{dX}{dx'} = 0$$

and forming the similar equations with the remaining variables and adding

$$A\delta a + B\delta b + C\delta c\,.. + 2F\delta f + 2G\delta g + 2H\delta h + ..$$
$$+ K\left(\frac{dX}{dx'} + \frac{dY}{dy'} + \frac{dZ}{dz'}..\right) = 0;$$

§ 5. **Lagrange's second form.**

Here the equations of motion are assumed to be

$$\frac{d}{dt}\frac{dT}{dx'} - \frac{dT}{dx} - P = 0,$$

$$\frac{d}{dt}\frac{dT}{dy'} - \frac{dT}{dy} - Q = 0,$$

$$\frac{d}{dt}\frac{dT}{dz'} - \frac{dT}{dz} - R = 0.$$

$$\vdots$$

where $2T$ represents the vis viva of the system, $x, y, z, \ldots$ are the independent variables on which the solution of the problem depends, and $x', y', z' \ldots$ their differential coefficients with respect to the time. It is assumed as before $P, Q, R \ldots$ do not contain $x', y', z' \ldots$

Suppose these equations give

$$dt : dx : dy : dz \ldots : dx' : dy' : dz' \ldots$$
$$= 1 : x' : y' : z' \ldots : X : Y : Z \ldots$$

Then the equation which determines the multiplier M takes as before the form

$$\delta M + M\left(\frac{dX}{dx'} + \frac{dY}{dy'} + \frac{dZ}{dz'} + \ldots\right). = 0.$$

To reduce this equation, substituting for T its value which is of the form

$$T = \tfrac{1}{2}.(ax'^2 + by'^2 + cz'^2 \ldots + 2fy'z' + 2gz'x' + 2hx'y' \ldots)$$

and putting for shortness

$$L = ax' + hy' + gz' \ldots$$
$$M = hx' + by' + fz' \ldots$$
$$N = gx' + fy' + cz' \ldots$$
$$\vdots$$

[redacted]

The equations which determine $X, Y, Z. \ldots$ are

$$aX + hY + gZ\ldots + \delta L - \frac{dT}{dx} - P = 0,$$

$$hX + bY + fZ\ldots + \delta M - \frac{dT}{dy} - Q = 0,$$

$$gX + fY + cZ\ldots + \delta N - \frac{dT}{dz} - R = 0.$$

$$\vdots$$

Whence, differentiating with respect to x',

$$a\frac{dX}{dx'} + h\frac{dY}{dx'} + g\frac{dZ}{dx'}\ldots + \delta a \qquad\qquad = 0,$$

$$h\frac{dX}{dx'} + b\frac{dY}{dx'} + f\frac{dZ}{dz'}\ldots + \delta h + \frac{dM}{dx} - \frac{dL}{dy} = 0,$$

$$g\frac{dX}{dx'} + f\frac{dZ}{dx'} + c\frac{dZ}{dx'}\ldots + \delta g + \frac{dN}{dx} - \frac{dL}{dz} = 0.$$

$$\vdots$$

Or representing by K the determinant formed with $a, h, g\ldots$ $h, b, f, \ldots g, f, c, \ldots$ and by $A, H, G, \ldots H, B, F, \ldots G, F, C\ldots$ the inverse system of coefficients, we have

$$K\frac{dX}{dx'} + A\delta a + H\delta h + G\delta g\ldots\ldots$$
$$+ \quad \bullet \quad + H\left(\frac{dM}{dx} - \frac{dL}{dy}\right) + G\left(\frac{dN}{dx} - \frac{dL}{dz}\right)\ldots = 0,$$

and similarly

$$K\frac{dY}{dy'} + H\delta h + B\delta b + F\delta f\ldots\ldots$$
$$+ H\left(\frac{dL}{dy} - \frac{dM}{dx}\right) + \quad \bullet \quad + F\left(\frac{dN}{dy} - \frac{dM}{dz}\right)\ldots = 0,$$

$$K\frac{dZ}{dz'} + G\delta g + F\delta f + C\delta c\ldots\ldots$$
$$+ G\left(\frac{dL}{dz} - \frac{dN}{dx}\right) + F\left(\frac{dM}{dz} - \frac{dN}{dy}\right) + \quad \bullet \quad \ldots = 0.$$

$$\vdots$$

Whence, adding,

$$K\left(\frac{dX}{dx'} + \frac{dY}{dy'} + \frac{dZ}{dz'}\ldots\right)$$
$$+ A\delta a + B\delta b + C\delta c\ldots + 2F\delta f + 2G\delta g + 2H\delta h\ldots = 0;$$

Or putting for shortness

$$\begin{cases} P - \dfrac{dT}{dx} = X, & \dfrac{dT}{d\xi} = \Xi, \\ Q - \dfrac{dT}{d\eta} = Y, & \dfrac{dT}{d\eta} = \mathrm{H}, \\ \vdots & \vdots \end{cases}$$

they become

$$dt : dx : dy : dz \,.\,. : d\xi : d\eta : d\zeta \,.\,.$$
$$= 1 : \Xi : \mathrm{H} : \Omega \,.\,. : X : Y : Z \,.\,.$$

and writing the equation in M under the form

$$\delta M + M \,.\left(\frac{d\Xi}{dx} + \frac{d\mathrm{H}}{dy} + \,.\,.\, + \frac{dX}{d\xi} + \frac{dY}{d\eta} + \,.\,.\right) = 0\,;$$

$$\left(\text{where } \delta = \frac{d}{dt} + \Xi\,\frac{d}{dx} + \mathrm{H}\,\frac{d}{dy} \,.\,. + X\,\frac{d}{d\xi} + Y\,\frac{d}{dy} + \,.\,.\right).$$

we see immediately that (P, Q. . being as before independent of the velocities, and consequently of ξ, η, ζ. .),

$$\frac{d\Xi}{dx}+\frac{dX}{d\xi}=0,\quad \frac{dH}{dy}+\frac{dY}{d\eta}=0,\ \&c.$$

Hence $\delta M=0$, which is satisfied by $M=1$.

58, *Chancery Lane, Feb.* 6, 1847.

ON A MULTIPLE INTEGRAL CONNECTED WITH THE THEORY OF ATTRACTIONS.

By Arthur Cayley.

Mr. Boole has given for the integral with (n) variables

$$V=\int\frac{\phi\left(\dfrac{x^2}{f^2}+\dfrac{y^2}{g^2}+\,.\,.\right)dx\,dy\,.\,.}{[(a-x)^2+(b-y)^2\,.\,.+u^2]^{\frac{1}{2}n+q}}\ \ldots\ldots(1);$$

limits $\qquad \dfrac{x^2}{f^2}+\dfrac{y^2}{g^2}+\,.\,.\leqq 1,$

the following formula, or one which may readily be reduced to that form,*

$$V=\frac{fg\,.\,.\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n+q)}\int_\eta^\infty\frac{Ss^{-q-1}ds}{\sqrt{\{(s+f^2)(s+g^2)\,.\,.}}\ \ldots(2),$$

where $\qquad S=\dfrac{(1-\sigma)^{-q}}{\Gamma(-q)}\displaystyle\int_0^1 t^{-q-1}\phi\{\sigma+t(1-\sigma)\}\,dt\ \ldots(3);$

in which $\qquad \sigma=\dfrac{a^2}{f^2+s}+\dfrac{b^2}{g^2+s}\,.\,.+\dfrac{u^2}{s}\ \ldots\ldots\ldots(4),$

and η is determined by

$$1=\frac{a^2}{f^2+\eta}+\frac{b^2}{g^2+\eta}\,.\,.+\frac{u^2}{\eta}.$$

Suppose $f=g=\ldots.=\infty$; also assume

$$\phi(\lambda)=\frac{1}{(f^2\lambda+v^2)^{\frac{1}{2}n+q'}}\ \ldots\ldots\ldots\ldots(5);$$

so that the integral becomes

$$U=\int\frac{dx\,dy\,.\,.}{(x^2+y^2\,.\,.+v^2)^{\frac{1}{2}n+q'}\{(x-a)^2+.+u^2\}^{\frac{1}{2}n+q}}\ \ldots(6),$$

the limits for each variable being $-\infty$, ∞.

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* See Note at the end of this paper.

$$\int_{-\infty}^{\infty}(4u^2v^2+x^2)^{q-\frac{1}{2}}e^{i\theta x}dx=\frac{1}{\Gamma\left(\frac{1}{2}-q\right)}\int_{-\infty}^{\infty}dx\int_0^{\infty}dt\; t^{-\frac{1}{2}-q}e^{-t(4u^2v^2+x^2)+i\theta x}$$

$$=\frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}-q\right)}\int_0^{\infty}dt\; t^{-1-q}e^{-4u^2v^2t-\frac{\theta^2}{4t}}.$$

Or, putting $4uv\ \sqrt{t}=\sqrt{(s+4uv)}\pm\sqrt{s}$ (which is a transformation already employed in the present paper), the formula required follows immediately.

Now, by a formula due to M. Catalan, but first rigorously demonstrated by M. Serret,

$$\int_0^{\infty}\frac{\cos ax\,dx}{(1+x^2)^n}=\frac{\pi}{(\Gamma n)^2}\int_0^{\infty}e^{-(a+2z)}(z+a)^{n-1}z^{n-1}dz,$$

(*Liouville*, tom. VIII., p. 1), and by a slight modification in the form of this equation

$$\int_{-\infty}^{\infty}(4u^2v^2+x^2)^{q-\frac{1}{2}}e^{i\theta x}dx=\frac{\pi e^{-2uv\theta}(4uv)^{2q}}{\theta^{2q}\Gamma^2\left(\frac{1}{2}-q\right)}\int_0^{\infty}s^{-q-\frac{1}{2}}(s+4uv)^{-q-\frac{1}{2}}e^{-\theta s}ds,$$

which, compared with (16), gives the required equation.

---

* A paper by M. Schlömilch "Note sur la Variation des Constantes Arbitraires d'une Integrale definie," *Crelle*, tom. XXXIII. p. 268-280, will be found to contain formulæ analogous to some of the preceding ones.

the upper sign from $s=\infty$ to $s=\dfrac{u}{v}$, and the lower one from $s=\dfrac{u}{v}$ to $s=0$, it is easy to derive

$$\Theta=(2v)^{2q}\int_0^\infty \frac{\{\sqrt{(s+4uv)}+\sqrt{s}\}^{-2q}+\{\sqrt{(s+4uv)}-\sqrt{s}\}^{-2q}}{\sqrt{s}\ \sqrt{(s+4uv)}}\,\phi(s+j+2uv)\,ds \qquad \ldots\ldots (10).$$

Now, by a formula which will presently be demonstrated,

$$\int_0^\infty \frac{\{\sqrt{(s+4uv)}+\sqrt{s}\}^{-2q}+\{\sqrt{(s+4uv)}-\sqrt{s}\}^{-2q}}{\sqrt{s}\ \sqrt{(s+4uv)}}\,e^{-\theta s}ds$$
$$=\frac{2\sqrt{\pi}}{\Gamma(\frac{1}{2}-q)}\,\theta^{-q}\int_0^\infty s^{-\frac{1}{2}-q}(s+4uv)^{-\frac{1}{2}-q}e^{-\theta s}ds \ \ldots\ldots (11);$$

whence $$\int_0^\infty \frac{\{\sqrt{(s+4uv)}+\sqrt{s}\}^{-2q}+\{\sqrt{(s+4uv)}-\sqrt{s}\}^{-2q}}{\sqrt{s}\ \sqrt{(s+4uv)}}\,Fs.ds$$
$$=\frac{2\sqrt{\pi}}{\Gamma(\frac{1}{2}-q)}\int_0^\infty s^{-\frac{1}{2}-q}(s+4uv)^{-\frac{1}{2}-q}\left(-\frac{d}{ds}\right)^{-q}Fs.ds \ \ldots\ldots (12).$$

So that by merely changing the function

$$\Theta=\frac{2^{2q+1}v^{2q}\sqrt{\pi}}{\Gamma(\frac{1}{2}-q)}\int_0^\infty s^{-\frac{1}{2}-q}(s+4uv)^{-\frac{1}{2}-q}\left(-\frac{d}{ds}\right)^{-q}\phi(s+j+2uv)\,ds \qquad \ldots\ldots (13);$$

and thence in the particular case in question

$$U=\frac{2^{2q+1}v^{2q}\pi^{\frac{1}{2}(n+1)}}{\Gamma(\frac{1}{2}-q)\,\Gamma(\frac{1}{2}n-q)}\int_0^\infty s^{-\frac{1}{2}-q}(s+4uv)^{-\frac{1}{2}-q}(s+j+2uv)^{-\frac{1}{2}n+q}ds \qquad \ldots\ldots (14),$$

by means of the formula

$$\left(-\frac{d}{ds}\right)^{-q}(s+a)^{-\frac{1}{2}n}=\frac{\Gamma(\frac{1}{2}n+q)}{\Gamma(\frac{1}{2}n)}(s+a)^{-\frac{1}{2}n+q}.$$

But as there may be some doubt about this formula, which is not exactly equivalent either to Liouville's or Peacock's expression for the general differential coefficient of a power, it is worth while to remark that, by first transforming the $\frac{1}{2}n^{\text{th}}$ power into an exponential, and then reducing as above, (thus avoiding the general differentiation), we should have obtained

$$U=\frac{2^{2q+1}v^{2q}\pi^{\frac{1}{2}(n+1)}}{\Gamma(\frac{1}{2}-q)\,\Gamma(\frac{1}{2}n+q)\,\Gamma(\frac{1}{2}n-q)}$$
$$\int_0^\infty d\theta\int_0^\infty ds\ \theta^{\frac{1}{2}n-q-1}e^{-\theta(s+j+2uv)}s^{-\frac{1}{2}-q}(s+4uv)^{-\frac{1}{2}-q}e^{-\theta s},$$

which reduces itself to the equation (14) by simply perform-

(
cients of the equation are imaginary quantities.

2. I must first explain the method which I adopt for representing the equation (1) when z is restricted to real values. We have

$$z = f(x + y\sqrt{-1})$$
$$= e^{y\sqrt{-1}\frac{d}{dx}} f(x) = \left(\cos y \frac{d}{dx} + \sqrt{-1} \sin y \frac{d}{dx}\right) f(x);$$

which, supposing the coefficients of $f(x)$ to be real, resolves itself into these two equations,

$$z = \left(\cos y \frac{d}{dx}\right) f(x) \ldots\ldots\ldots\ldots(2),$$

$$0 = \left(\sin y \frac{d}{dx}\right) f(x) \ldots\ldots\ldots\ldots(3).$$

The expressions $\cos y \frac{d}{dx}$ and $\sin y \frac{d}{dx}$ are to be supposed expanded in powers of $y \frac{d}{dx}$, and when the differentiations indicated are performed, the equations (2) and (3) will consist of only a finite number of terms.

3. If $P = \left(\cos y \frac{d}{dx}\right) f(x)$ and $Q = \left(\sin y \frac{d}{dx}\right) f(x)$, it is easy to see that

$$\frac{dP}{dx} = \left(\cos y \frac{d}{dx}\right) f'(x), \quad \frac{dP}{dy} = -\left(\sin y \frac{d}{dx}\right) f'(x),$$

$$\frac{dQ}{dx} = \left(\sin y \frac{d}{dx}\right) f'(x), \quad \frac{dQ}{dy} = \left(\cos y \frac{d}{dx}\right) f'(x),$$

$$\frac{d^2P}{dx^2} = \left(\cos y \frac{d}{dx}\right) f''(x), \quad \frac{d^2P}{dx\,dy} = -\left(\sin y \frac{d}{dx}\right) f''(x),$$

$$\frac{d^2P}{dy^2} = -\left(\cos y \frac{d}{dx}\right) f''(x),$$

$$\frac{d^2Q}{dx^2} = \left(\sin y \frac{d}{dx}\right) f''(x), \quad \frac{d^2Q}{dx\,dy} = \left(\cos y \frac{d}{dx}\right) f''(x),$$

$$\frac{d^2Q}{dy^2} = -\left(\sin y \frac{d}{dx}\right) f''(x),$$

&c. &c.

$$\text{therefore } \frac{dP}{dx} = \frac{dQ}{dy},$$

$$\frac{dQ}{dx} = -\frac{dP}{dy},$$

$$\frac{d^2P}{dx^2} = \frac{d^2Q}{dx\,dy} = -\frac{d^2P}{dy^2},$$

$$\frac{d^2Q}{dx^2} = -\frac{d^2P}{dx\,dy} = -\frac{d^2Q}{dy^2},$$

&c. &c.

4. THEOREM:—*If $f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots + p_n$, where $p_1, p_2 \ldots p_n$, are either real or imaginary, then if x be allowed to assume all values real or imaginary, subject to the condition that $f(x)$ is real, $f(x)$ does not admit of a maximum or minimum value.*

5. For simplicity's sake suppose first the coefficients $p_1, p_2 \ldots$ to be real. Then putting for x, $x + y\sqrt{-1}$, the equation $z = f(x)$ resolves itself into these two:

$$z = \left(\cos y \frac{d}{dx}\right) f(x) = P,$$

$$0 = \left(\sin y \frac{d}{dx}\right) f(x) = Q.$$

$$\left.\begin{aligned} 1.2.\delta^2 z &= \frac{d^2P}{dx^2}\,\delta x^2 + 2\,\frac{d^2P}{dx\,dy}\,\delta x\,\delta y + \frac{d^2P}{dy^2}\,\delta y^2 \\ \text{and}\quad 0 &= \frac{d^2Q}{dx^2}\,\delta x^2 + 2\,\frac{d^2Q}{dx\,dy}\,\delta x\,\delta y + \frac{d^2Q}{dy^2}\,\delta y^2 \end{aligned}\right\} \ldots (6),$$

(since the coefficients of the quantity $\delta^2 y$ are zero).

Let A and B be the values assumed by $\frac{d^2P}{dx^2}$ and $\frac{d^2Q}{dx^2}$, corresponding to the values of x and y given by equations (5). Then, by the relations established in Art 3,

$$1.2.\delta^2 z = A\delta x^2 - 2B\,\delta x\,\delta y - A\delta y^2,$$

$$0 = B\delta x^2 + 2A\,\delta x\,\delta y - B\delta y^2.$$

Let $\qquad \delta x = \epsilon \cos\phi,$

$\qquad \delta y = \epsilon \sin\phi;$

therefore $1.2\,\delta^2 z = \epsilon^2\left\{A\cos 2\phi - B\sin 2\phi\right\},$

$\qquad 0 = \epsilon^2\left\{B\cos 2\phi + A\sin 2\phi\right\},$

which equations may be put under the form

$$1.2.\delta^2 z = \epsilon^2 C\cos(2\phi + \alpha),$$

$$0 = \sin(2\phi + \alpha);$$

therefore $\qquad 2\phi + \alpha = 0 \text{ or } \pi,$

and $\qquad 1.2\,\delta^2 z = \pm\,\epsilon^2 C.$

Hence $\delta^2 z$ has two values, one positive and the other negative, and therefore the value of z corresponding to the values of x and y, supposed to be obtained, cannot be said to be either a maximum or a minimum.

6. But it is possible that $\delta^2 z$ may vanish; we shall therefore consider the general case in which $\delta^m z$ is the first of the series of quantities δz, $\delta^2 z$, $\delta^3 z$, &c., which does not vanish; and it is not difficult to see that in this case we have

$$1.2\ldots m\,\delta^m z = \left(\delta x\frac{d}{dx} + \delta y\frac{d}{dy}\right)^m P,$$

$$0 = \left(\delta x\frac{d}{dx} + \delta y\frac{d}{dy}\right)^m Q.$$

Let A, B be the the values assumed by $\dfrac{d^m P}{dx^m}$, $\dfrac{d^m Q}{dx^m}$, respectively; then, in consequence of the relations of Art. 3, we shall have

$$1.2\ldots m\delta^m z = A\delta x^m - mB\delta x^{m-1}\delta y - \frac{m.(m-1)}{1.2}A\delta x^{m-2}\delta y^2 + \ldots$$

$$0 = B\delta x^m + mA\delta x^{m-1}\delta y - \frac{m.(m-1)}{1.2}B\delta x^{m-2}\delta y^2 - \ldots$$

Let $\delta x = \epsilon\cos\phi$, $\delta y = \epsilon\sin\phi$; therefore

$$1.2\ldots m\delta^m z = \epsilon^m\left\{A\left(\cos^m\phi - \frac{m.(m-1)}{1.2}\cos^{m-2}\phi\sin^2\phi + \ldots\right)\right.$$
$$\left. - B\,(m\cos^{m-1}\phi\sin\phi\ldots)\right\},$$

$$0 = \epsilon^m\left\{B\left(\cos^m\phi - \frac{m.(m-1)}{1.2}\cos^{m-2}\phi\sin^2\phi + \ldots\right)\right.$$
$$\left. + A\,(m\cos^{m-1}\phi\sin\phi\ldots)\right\},$$

or
$$1.2\ldots m\delta^m z = \epsilon^m\{A\cos m\phi - B\sin m\phi\},$$
$$0 = \epsilon^m\{B\cos m\phi + B\sin m\phi\};$$

which may be put under the form

$$1.2\ldots m\delta^m z = \epsilon^m C\cos(m\phi + a),$$
$$0 = \sin(m\phi + a);$$

therefore $m\phi + a = k\pi$, where k may have any one of the values $0, 1, 2\ldots\ldots m-1$, and

$$1.2\ldots m\delta^m z = (-1)^k\epsilon^m C.$$

Therefore $\delta^m z$ has m values which are alternately positive and negative, and therefore z admits of no maximum or minimum value.

Let

therefore

$$\therefore f(x+y\sqrt{-1}) = \left(\cos y\frac{d}{dx} + \sqrt{-1}\sin y\frac{d}{dx}\right)M$$
$$+\left(\cos y\frac{d}{dx} + \sqrt{-1}\sin y\frac{d}{dx}\right)N\sqrt{-1}$$
$$=\left(\cos y\frac{d}{dx}\right)M - \left(\sin y\frac{d}{dx}\right)N$$
$$+\sqrt{-1}\left\{\left(\sin y\frac{d}{dx}\right)M + \left(\cos y\frac{d}{dx}\right)N\right\}.$$

Hence if $\quad f(x+y\sqrt{-1}) = P + Q\sqrt{-1},$

we have $\quad z = P = \left(\cos y\frac{d}{dx}\right)M - \left(\sin y\frac{d}{dx}\right)N,$

$$0 = Q = \left(\sin y\frac{d}{dx}\right)M + \left(\cos y\frac{d}{dx}\right)N.$$

Differentiating

$$\frac{dP}{dx} = \left(\cos y\frac{d}{dx}\right)\frac{dM}{dx} - \left(\sin y\frac{d}{dx}\right)\frac{dN}{dx},$$

$$\frac{dP}{dy} = -\left(\sin y\frac{d}{dx}\right)\frac{dM}{dx} - \left(\cos y\frac{d}{dx}\right)\frac{dN}{dx},$$

$$\frac{dQ}{dx} = \left(\sin y\frac{d}{dx}\right)\frac{dM}{dx} + \left(\cos y\frac{d}{dx}\right)\frac{dN}{dx},$$

$$\frac{dQ}{dy} = \left(\cos y\frac{d}{dx}\right)\frac{dM}{dx} - \left(\sin y\frac{d}{dx}\right)\frac{dN}{dx},$$

$$\therefore \frac{dP}{dx} = \frac{dQ}{dy}, \quad \frac{dP}{dy} = -\frac{dQ}{dx};$$

and in like manner we should find that

$$\frac{d^2P}{dx^2} = \frac{d^2Q}{dx\,dy} = -\frac{d^2P}{dy^2},$$

$$\frac{d^2Q}{dx^2} = -\frac{d^2P}{dx\,dy} = -\frac{d^2Q}{dy^2},$$

and so on, as in Art. 3. Hence the relations between the differential coefficients of P and Q being the same as in the case of the quantities $p_1, p_2 \ldots$ being real, the investigation given in that case is equally applicable to the more general one now under consideration.

Hence the theorem enunciated is true.

8. Since then $f(x)$ admits of no maximum or minimum value, we may give x a succession of values of the form $x + y\sqrt{-1}$, which shall cause $f(x)$ to assume all real values intermediate to $+\infty$ and $-\infty$. Let us now examine how many such sets of values can be found.

When x is very large, the equation

$$z = x^n + p_1 x^{n-1} + \ldots$$

degenerates into the following,

$$z = x^n.$$

Let $\qquad x = \rho(\cos\theta + \sqrt{-1}\sin n\theta),$

therefore $\qquad z = \rho^n(\cos n\theta + \sqrt{-1}\sin\theta);$

which is equivalent to these two,

$$z = \rho^n \cos n\theta,$$

$$0 = \sin n\theta;$$

therefore $n\theta = k\pi$, where k may have any one of the values $0, 1, 2 \ldots (2n-1)$, and

$$z = (-1)^k \rho^n.$$

Hence, when $k = 0, 2, 4 \ldots$, z becomes $+\infty$ when ρ is indefinitely increased, and when $k = 1, 3, 5 \ldots$, becomes $-\infty$; and therefore there are n series of values which may be assigned to x, which will make z vary continuously from $+\infty$ and $-\infty$, and therefore n values may be found which will make z vanish, that is, the equation $f(x) = 0$ has n roots.

Cambridge, *Feb.* 10, 1847.

[illegible] which the magnetized sphere may be supposed to be [illegible], it is easily shown, as has also been done by Poisson, that the components of the resulting force on the sphere are given by the equations

$$X' = \frac{dX}{dx}\cos\lambda + \frac{dX}{dy}\cos\mu + \frac{dX}{dz}\cos\nu,$$
$$Y' = \frac{dY}{dx}\cos\lambda + \frac{dY}{dy}\cos\mu + \frac{dY}{dz}\cos\nu,$$
$$Z' = \frac{dZ}{dx}\cos\lambda + \frac{dZ}{dy}\cos\mu + \frac{dZ}{dz}\cos\nu,$$

[illegible] is the volume of the sphere, and λ, μ, ν the cosines of the angles made by the direction of magnetization with the axes. Now since the direction is that of the force R, we have

$$\cos\lambda = \frac{X}{R}, \quad \cos\mu = \frac{Y}{R}, \quad \cos\nu = \frac{Z}{R}.$$

[illegible] we have

$$X' = [illegible]\left(X\frac{dX}{dx} + Y\frac{dX}{dy} + Z\frac{dX}{dz}\right)$$
$$Y' = [illegible]\left(X\frac{dY}{dx} + Y\frac{dY}{dy} + Z\frac{dY}{dz}\right) \quad \ldots\ldots \; (2)$$
$$Z' = [illegible]\left(X\frac{dZ}{dx} + Y\frac{dZ}{dy} + Z\frac{dZ}{dz}\right)$$

[illegible] magnet, or a closed galvanic circuit [illegible] necessarily a complete differential, [illegible]

$$\frac{dZ}{dy} = \frac{dY}{dz}, \quad \frac{dZ}{dx} = \frac{dX}{dz}, \quad \frac{dX}{dy} = \frac{dY}{dx} \quad \ldots\ldots \; (3)$$

[illegible] (2) by means of these equations, we find

$$F = \frac{4\pi}{3}\sigma\left(X\frac{dX}{dx} + Y\frac{dY}{dx} + Z\frac{dZ}{dx}\right) = \frac{4\pi}{3}\sigma R\frac{dR}{dx}$$
$$G = \frac{4\pi}{3}\sigma\left(X\frac{dX}{dy} + Y\frac{dY}{dy} + Z\frac{dZ}{dy}\right) = \frac{4\pi}{3}\sigma R\frac{dR}{dy} \quad \ldots \; (4)$$
$$H = \frac{4\pi}{3}\sigma\left(X\frac{dX}{dz} + Y\frac{dY}{dz} + Z\frac{dZ}{dz}\right) = \frac{4\pi}{3}\sigma R\frac{dR}{dz}$$

is one below the lower end (at a distance $= \frac{\quad}{\sqrt{2}}$, if a, the radius of the cylinder, be very great compared with its thickness, and very small compared with its length, and if the distribution of magnetism be uniform) at which the resultant force is a maximum. If, on moving a small diamagnetic sphere upwards from this position, we arrive at a point where the force urging it upwards is greater than the weight, and then let it move freely from rest, it will oscillate about a position of stable equilibrium. It will probably be impossible ever to observe this phenomenon, on account of the difficulty of getting a magnet strong enough, and a diamagnetic substance sufficiently light, as the forces manifested in all cases of diamagnetic induction hitherto examined are excessively feeble.

11. A very curious phenomenon might readily be observed, according to the results given above, by placing two bar-magnets, with similar poles in the neighbourhood of

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* *Experimental Researches*, § 2418.

† The law of induction in a mass of any form, whether of magnetic or diamagnetic matter, may be stated as follows. Let R be the magnetic force upon a point within an infinitely small spherical surface, described round a point P in the mass, resulting from the magnetism of all the matter external to this surface. The intensity of the magnetism at P is equal to $\frac{3}{4}\pi i\, R$, and the direction is that of the resultant force R.

or *Spence's* integral, while virtually treating of the same under the form $\int(1+x)^{-1}\log(\pm x).dx$. It appears moreover from Kummer (p. 220), that *Clausen* has actually tabulated my integral ϟ in p. 298 of *Crelle's Journal*, Vol. VIII., under the form $-\int_0 \log(\pm 2\sin\frac{1}{2}a)\,da$.

Professor Kummer conceives of the general integral under the form $\int F_1x\int F_2x\,dx.dx$; and he has also extended his views to the third, fourth, fifth, &c. orders of rational integrals (for this appears to be the more appropriate title), and has exhibited in them integrals which are analogous to those of the second order.

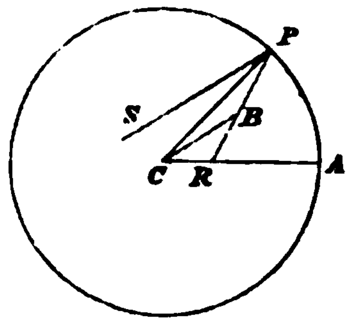
F. W. NEWMAN.

May 6th, 1847.

II. *On the Caustic by Reflection at a Circle.*

[*To the Editor.*]

A paper by Mr. Cayley, under the above title, having been published in the last number of your *Journal*, it appears to me that both M. de St. Laurent and Mr. Cayley have overlooked the admirably symmetrical solution of the problem given by Lagrange in the *Mem. de Turin*. Thinking that some of your correspondents may be interested in it, I beg to send you a translation.



Let B be the luminous point, RBP an incident, and PS a reflected ray; CA a fixed radius $ACP = \alpha$, $ACB = \epsilon$, reciprocal of $CB = c$, reciprocal of $CP = a$. The equations of the incident and reflected ray where $u = \dfrac{1}{r}$ may be written

$$u = A \sin \theta + B \cos \theta, \text{ incident ray,}$$

$$u = A \sin (2\alpha - \theta) + B \cos (2\alpha - \theta), \text{ reflected,}$$

the conditions for determining A and B being

$$a = A \sin \alpha + B \cos \alpha,$$

$$c = A \sin \epsilon + B \cos \epsilon;$$

whence $A = \dfrac{a \cos \epsilon - c \cos \alpha}{\sin (\alpha - \epsilon)}$, $B = \dfrac{c \sin \alpha - a \sin \epsilon}{\sin (\alpha - \epsilon)}$.

Substituting these values, the equation of the reflected ray becomes

$$a \sin (2\alpha - \theta - \epsilon) = u \sin (\alpha - \epsilon) + c \sin (\alpha - \theta);$$

from which and its differential with respect to the arbitrary parameter α, the equation to the caustic or envelope of the reflected rays will be found by eliminating α.

In this, α being the only quantity treated as variable in the differentiation, let $2\alpha - \theta - \epsilon = 2\phi$,

therefore $$\alpha = \phi + \tfrac{1}{2}(\theta + \epsilon),$$

and the equation becomes

$$a \sin 2\phi = u \sin \{\phi + \tfrac{1}{2}(\theta - \epsilon)\} + c \sin \{\phi - \tfrac{1}{2}(\theta - \epsilon)\}.$$

Make $$P = \frac{(u + c) \cos \frac{1}{2}(\theta - \epsilon)}{2a},$$

$$Q = \frac{(u - c) \sin \frac{1}{2}(\theta - \epsilon)}{2a}.$$

Also $$x = \frac{1}{\cos \phi}, \quad y = \frac{1}{\sin \phi},$$

and the equation becomes

$$Px + Qy = 1,$$

with the condition $$x^{-2} + y^{-2} = 1.$$

Hence $$P = \lambda x^{-3},$$

$$Q = \lambda y^{-3}.$$

Multiplying by x and y, and adding, we find $\lambda = 1$;

therefore $$x^{-2} = P^{\frac{2}{3}}, \quad y^{-2} = Q^{\frac{2}{3}}.$$

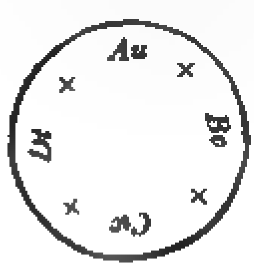
For this will be an equation of the second order, since u, v, w, t are linear in x and y. This equation is also satisfied by $u = 0,\ t = 0$; $u = 0,\ v = 0$; $v = 0,\ w = 0$; $w = 0,\ t = 0$; and therefore the curve passes through $ABCD$. Moreover the general form $ax + by + c$ being taken for each of the functions u, v, w, t, we may conceive each to have been multiplied by an arbitrary constant previously to combination in (1), and therefore the equation (1) will have all the generality possible.

Now let A, B, C, D be four constants such that

$$Au + Bv + Cw + Dt = 0 \text{ identically} \ldots\ldots (2).$$

Then the equation $Au + Cw = 0$ is the same as $Bv + Dt = 0$. But $Au + Cw = 0$ represents a line through F, and $Bv + Dt = 0$ one through E, and hence either equation represents EF.

Arrange the terms of (2) in circular order, thus



[illegible]

[illegible]

each into as many detached systems as there are separate belts, will be distributed in a manner exactly similar.

Such is the case when both bounding curves consist of the same number of detached curves: when however the number is not the same for both, then will only some of the detached systems of lines fall under the head already considered, and the others must be referred to some one or other of the following cases.

In the case when one of the two curves on each sheet is imaginary and the other real and single, then obviously the modification from the above general type will be, that one of the two systems of lines will have no cusps and the other will have no envelope, but in other respects their nature and their distribution will be exactly similar, the point or points of contact of each line of the enveloped system, whatever that may happen to be, with the single real curve envelope of that system coinciding always with the cusp or cusps of the corresponding line of the other system, and the two corresponding tangents at every coincident point being always conjugate to each other with respect to the surface; which indeed also is always the case all over the available portions of the envelope where a line of either system has a point in common with or, which is the same thing, intersects a line of the conjugate system.

But, as is much more frequently the case, one of the bounding curves being imaginary, when the one real curve on each sheet consists of two detached curves or of two or more distinct pairs of detached curves, then, for the reasons before explained, will the available intervals between each pair be all continuous and generally, of finite breadth, though returning back into themselves or going off to infinity indifferently as the case may be; the two systems of lines will be divided both into as many detached portions as there are belts of this nature, that is, as there are pairs of curves, and of the portions on each belt every line of one system will touch the two bounding curves of that belt each at least in one point, but often in more, and sometimes in an infinite number, but always both in the same number of points; and every line of the other system will have as many pairs of cusps as the lines of the former have pairs of points of contact, every pair of cuspal points of every line of that system lying on the double curve and coinciding with the corresponding pairs of points of contact of the corresponding line of the former system, and the two corresponding tangents at every individual point of coincidence, being always

envelope be both altogether available, and the two systems of lines on each, the lines of contact and the lines of regression, will both be altogether changed in character, neither system will admit of an envelope, and no line of either system will be endowed with a cusp, but both systems of lines will consequently cover the whole surface, and will either or both on each sheet, as the case may be, either return back into themselves or extend in both directions and meet at infinity; and at every point taken arbitrarily on either sheet there will cross two distinct lines of each system intersecting at an angle of finite magnitude, the two tangents to the two intersecting lines of either system being respectively the two conjugates with respect to the surface of the two tangents to the two corresponding intersecting lines of the other system.

This last property is generally true, whatever be the nature of the envelope and of the two bounding curves, all over the available regions of that surface; for at all the points of these regions there will obviously cross as many lines of regression

there are tangents which are principal axes, such tangents

giving the directions in which the lines of that system, whatever be their number, diverge from each point: but the cone of principal axes diverging from every point being always of the second order, there will generally diverge from each point two, and there can diverge but two tangents which will be principal axes, and therefore two and but two lines of regression, and consequently also two and but two lines of the conjugate system. Hence, to state the property before us in its greatest generality, we may say, that through every point on each sheet of every surface envelope of a system of principal axes subject to a single condition, including not only the available but also the untouched region or regions of that surface and the separating curve or curves, there pass always two lines of each species, of regression and of contact, the pairs being simultaneously both real, both imaginary, or both coincident, as the case may be.

It not unfrequently happens, especially on envelopes of which the two sheets are altogether available, that points exist for which the two tangent principal axes are conjugate to each other with respect to the surface: at all such points the two diverging lines of each species coincide obviously in direction with each other, and therefore if any envelope be such that the same takes place at every point, then for that surface will the two systems of lines of contact and of regression absolutely coincide with each other, that is, the same lines which form the curves of contact of one of the two component systems of developable surfaces, into which the enveloped system of axes may be resolved, will also form the curves of regression of the other component developable system; and conversely, if either sheet of an envelope possess the property that its two conjugate systems of lines coincide with each other, then at every point of that sheet will the two tangent principal axes be always conjugate to each other with respect to the surface. We shall see as we proceed, that there exists in every body a very extensive class of systems of envelopes, the class containing an infinite number of systems, and each system containing an infinite number of envelopes, all possessing these unusual properties, that both their sheets are entirely available, and that their two systems of lines on each sheet, of contact and of regression, coincide with each other all over the whole extent of the surface. Moreover, if on a surface of this nature, besides crossing at every point of the surface in directions which are always conjugate to each other, the two intersecting sets of lines of the same species intersect everywhere two and two at right

IV. *On a System of Magnetic*

Let λ be the potential produced by a m
about an axis OX at a point $P(x, y)$. Th
or lines of force, being the orthogonal
surfaces for which the potential is constan
through OX; and the system in the plan
orthogonal trajectory of the system of
Their equation, as was shewn in a paper '
of Motion of Heat referred to Curvilinear
IV. p. 40), is

$$\int y \left(\frac{d\lambda}{dy} dx - \frac{d\lambda}{dx} dy \right) = C$$

As an example, let λ be due to two st
in the line OX, at points M, M'; so that

$$\lambda = \frac{\mu(x-f)}{\{(x-f)^2+y^2\}^{\frac{3}{2}}} + \frac{\mu'(x-f')}{\{(x-f')^2+y^2\}^{\frac{3}{2}}} = \frac{\mu(\ldots}{\ldots}$$

By integration we find, from (1),

$$\frac{\mu y^2}{\Delta^3} + \frac{\mu' y^2}{\Delta'^3} = C,$$

for the equation of the system of magnetic curves.

If we take as a particular case, $C = 0$, we find $y^2 = 0$, which shows that the axis is a line of force; we have also, for another branch, corresponding to the same value of C,

$$\frac{\mu}{\Delta^3} + \frac{\mu'}{\Delta'^3} = 0.$$

As Δ and Δ' are essentially positive in the physical problem, this can only be satisfied if μ and μ' have different signs. For instance, if $\mu = 1$, $\mu' = -m$, we have

$$\Delta' = m^{\frac{1}{3}} \Delta.$$

The locus of this equation is, as is well known, a circle, which may be described thus. Divide MM' in A, and produce it to A_1, so that

$$M'A = m^{\frac{1}{3}}.MA \text{ and } M'A_1 = m^{\frac{1}{3}}.MA_1;$$

on AA_1 as diameter describe a circle.

This result was suggested to me by the solution of a corresponding problem (of much greater interest however) in fluid motion, verbally communicated to me by Mr. Stokes.

WILLIAM THOMSON.

St. Peter's College, May 18, 1847.

Indeed, whatever has been said in the present article respecting a system of principal axes subject to a single restricting condition, holds more generally and with scarcely an exception for every system of right lines in space subject to two independent restricting conditions, the only difference between the particular and the more general case being, that in the former one of the two restricting conditions is given and is always the same, while in the general case they are both variable and arbitrary. It was for this reason that we have dwelt so long on this (which is far from being the most interesting) part of our subject, because that we have all along been implicitly discussing the more general question respecting the management and properties of a system of right lines subject to two conditions, the nature and properties of the different systems of rule surfaces into which such a system of lines may be resolved, the nature and varieties of the surface, their envelope, and the consequent nature, position, distribution, and varieties of the two conjugate systems of generating curves on each sheet of that surface, the lines, namely of contact and of regression of the two component systems of developable surfaces into which every such system of right lines in space may be always resolved.

One property indeed (and it seems to be the only one) requires a different method of establishment, viz. that at every point of the envelope two and but two right lines of the system enveloped can touch that surface, and therefore that through every point on the same there can pass two and but two lines of each system of contact and of regression; for it is not every pair of conditions for which the cone resulting from one or either of them will be always of the second order, and besides that the property itself is not without exception true; that it holds however in the general case also, and holds moreover for the most part though not universally, may be easily shewn as follows: Let a system of right lines subjected to one of the two conditions, whatever they be, be constrained to pass all through a point, they will generate a cone of some order or other, let then another system of right lines subjected to the other condition be constrained to pass all through the same point, they will generate another cone: the intersecting sides of these two cones will evidently be the only lines passing through the point which fulfil at once the two conditions, and it is obvious that in general no three of them, except accidentally or in particular cases, will ever lie in the same plane. Hence we see that even one

Suppose that one of the two curves in any plane whose common tangents were the only lines in that plane, fulfilling at once the two restricting conditions had a double point, nodal or conjugate, or more generally a multiple point of any order whatever, then would all the tangents drawn from the multiple point to the other curve come under the head of common tangents to the two curves, and therefore in such cases exceptions would exist to the general rule that no more than two right lines of the system fulfilling the two conditions could at the same time pass through the same point and lie in the same plane. A very general and extensive class of exceptions of this nature is to be found in a system of right lines subject to the two independent conditions of touching two given surfaces; for if we draw any plane whatever intersecting both surfaces in a pair of curves, then will the common tangents to these curves be the lines in that plane which fulfil the two conditions; but in the particular case when the common intersecting plane touches either of the surfaces, then will the point of contact be a double point of the curve in which it intersects that surface, and obviously all tangents from that point to the other curve will fulfil the two conditions. The same manifestly may be said also of every system of right lines subject to the two conditions of having double contact with a single given surface, such being in fact but particular cases of the former, the two given surfaces being conceived as coming together and coinciding in one. Hence in the extensive class of cases where the complete envelope itself is given and where the restricting conditions are in contact with both its sheets, or double contact with its single sheet if it consist of only one, the above general property fails, and more than two tangents at the different points of the surface could in general be drawn fulfilling the two conditions.

In every case when we have a system of principal axes, such as we have been considering, subject to a single condition, the corresponding system of principal planes will, obviously, also envelope a surface; and to find that envelope when the restricting condition is given, we may proceed exactly on the principles already described—for the introduction of an arbitrary condition resolves the system of axes into a multitude of groups each forming a surface gauche or developable as the case may be, and therefore divides the whole corresponding system of planes into an infinite number of smaller systems having each for its envelope a developable surface, and of this system of developables the envelope is of

[redacted]

In order to specify the varieties of this general case, one or two remarks may be usefully appended. Let us then suppose the plane ACB to be horizontal, and CD to be the part of the line *above* ACB; whilst the part *below* ACB is denoted by Cd. Also, let Cq, Cp be equal to CQ, CP. Then,

(α). The angles made by inflecting lines from A and B to points in CD continually increase as the point to which the lines are inflected approach towards Q from D. After passing Q, the angles continually diminish till the point of inflection arrives at C. The angles then again increase till the point arrives at q; and, finally, in passing farther downwards the angles diminish incessantly. The extreme limits of the magnitudes of the angles in both directions is zero.

The problem, then, in this case has two maxima solutions and one minimum—using these terms in their modern mathematical sense.

(β). If a circle described through A, B, C, and cut the line EC produced in P′; and if P be constructed so that EP shall be equal to EP′: then the orthographic projection ACB of the angle APB will be equal to the angle APB itself.

(γ). If a circle be described through A, B to cut CE produced between C and Q′, in H′, and again in K′; and points H, K corresponding to them be taken in CD (these points are omitted in the figure to prevent confusion): then the angles AHB, AKB will be equal to one another. Whence two equal angles situated in different planes passing through AB may have the same orthographic projection. The same is obviously true of two angles AkB, AhB on the other side (or *below*) the plane ACB.

(δ). If the circle through A, B touching CE, touch it in C, the point Q coincides with C; and the angle ACB will be the greatest possible.

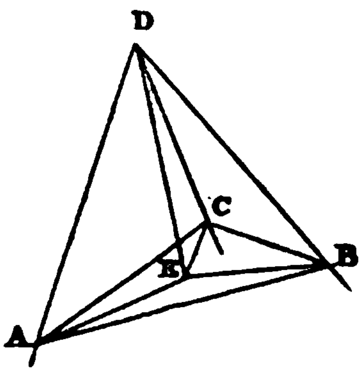
In this case, then, the ordinary assumption is correct as to the projection being greater than the projected angle. It is this which gives rise to the limitation expressed in the construction of the second case.

(ϵ). If the circle through A, B touch CE between C and c (c taken equidistant from E with C) the point determined by it is excluded from consideration; since no corresponding point could be constructed in CD, CE being the shortest line that can be drawn from E to CD. In this case, then, the usual assumption is also accurate.

EUCLID XI. 21. ***Every solid angle is contained by plane angles, which are together less than four right angles.***

The demonstration of this theorem will only require the property established in the *first case* of the preceding proposition together with the preceding propositions of Euclid. It will be divided into two cases corresponding to Euclid's own division.

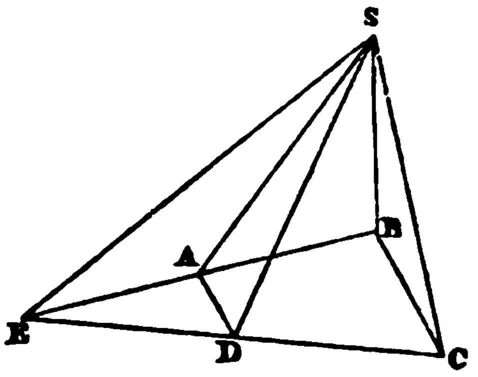
(α). Let the angle D be trihedral; take DA, DB, DC all equal; draw the perpendicular DE to the plane ABC; and join EA, EB, EC.



Then, since DEA, DEB, DEC are right angles, it readily follows from (I. 47) and the conditions of our construction, that AE, BE, CE are equal, and hence that E is the centre of the circle about ABC. Whence the perpendiculars from E to the sides of the triangle fall *between* the extremities of those lines, since those perpendiculars bisect the sides.

Wherefore ADB, BDC, CDA are respectively less than AEB, BEC, CEA; and hence their sum less than the four right angles to which AEB, BEC, CEA are equal.

(β). Let the angle S be tetrahedral, and produce two noncontiguous faces to meet in SE; also, let the system of planes be cut by any other plane, ABCDE.



Then, by the preceding case, the angles ESB, BSC, CSE are less than four right angles. But (XI. 20) the angle ASD is less than the two ESA, ESD; and hence the angles ASB, BSC, CSD, DSA are less than ESB, BSC, CSE; and therefore, *à fortiori*, less than four right angles.

(γ). In all other cases the sum of the plane angles may be shewn to become less and less as their number is increased; and hence, that the proposition is true in its most general form, as long as the dihedral angles are salient.

Royal Military Academy, Woolwich,
April 3, 1847.

$$= k^2 sn^2 u - \frac{1}{sn^2 u};$$

whence also

$$\frac{d^2}{du^2} \log snu = k^2 sn^2 u - k^2 sn^2 . (u + iK').$$

If for a moment

$$\psi_{,} u = \int_0 du sn^2 u, \qquad \psi_{,,} u = \int_0 du \int_0 du sn^2 u,$$

$$\log snu = k^2 \psi_{,,} u - k^2 \psi_{,,} (u + iK') + Au + B.$$

Or writing $(-u)$ for u and subtracting, $\psi_{,} u$ being an even function,

$$2Au = \pi i - k^2 \psi_{,,} (iK' - u) + k^2 \psi_{,,} (iK' + u),$$

or putting $u = K$,

$$2AK = \pi i - k^2 \psi_{,,} (iK' - K) + k^2 \psi_{,,} (iK' + K).$$

Now $\qquad sn^2 (u + K) - sn^2 (u - K) = 0,$

and therefore $\qquad \psi_{,} (u + K) - \psi_{,} (u - K) = 2\psi_{,} K,$

$$\psi_{,,} (u + K) - \psi_{,,} (u - K) = 2u\psi_{,} K;$$

or $\qquad \psi_{,,} (iK' + K) - \psi_{,,} (iK' - K) = 2iK' \psi_{,} K.$

Also $$E(u) = u - k^2\psi_{,}u,$$

or $$E = K - k^2\psi_{,}K, \quad \text{i.e. } \psi_{,}K = \frac{K}{k^2}\left(1 - \frac{E}{K}\right).$$

Hence $$A = iK'\left(1 - \frac{E}{K}\right) + \frac{\pi i}{2K},$$

$$\log sn u = k^2\psi_{,,}u - k^2\psi_{,,}(u + iK') + uiK'\left(1 - \frac{E}{K}\right) + \frac{\pi u i}{2K} + B$$

$$= k^2\psi_{,,}u - k^2\psi_{,,}(u + iK') + \tfrac{1}{2}\left[(u + iK')^2 - u^2\right]\left(1 - \frac{E}{K}\right) + \frac{\pi u i}{2K} + B',$$

i.e. $$\log sn u = \log \Theta(u + iK') - \log \Theta u + \frac{\pi u i}{2K} + B',$$

or, changing the constant,

$$sn u = Ce^{\frac{\pi u i}{2K}} \frac{\Theta(u + iK')}{\Theta u}.$$

Now, to determine C, write $u - iK'$ for u; this gives

$$\frac{1}{ksn u} = Ce^{\frac{\pi i}{2K}(u - iK')} \frac{\Theta u}{\Theta(u - iK')}:$$

and again changing (u) into $(-u)$,

$$-sn u = Ce^{-\frac{\pi u i}{2K}} \frac{\Theta(u - iK')}{\Theta u}:$$

whence, multiplying these last two equations,

$$C^2 = -\frac{1}{k} e^{-\frac{\pi K'}{2K}},$$

or $$C = \frac{1}{i\sqrt{k}} e^{-\frac{\pi K'}{4K}};$$

whence $$sn u = \frac{1}{i\sqrt{k}} e^{-\frac{\pi(K' - 2iu)}{4K}} \frac{\Theta(u + iK')}{\Theta u},$$

i.e. $$\sqrt{k}\, sn u = \frac{\mathrm{H}(u)}{\Theta(u)} \ldots\ldots\ldots\ldots (3);$$

and the equations (1), (2) and (3) may be considered as comprehending the theory of the functions H (u), $\Theta(u)$. The preceding process is, in fact, the converse of that made use of in the *Fund. Nova;* Jacobi having obtained for sn u an expression in the form of a fraction, takes the numerator of it for H (u) and the denominator for $\Theta(u)$, and thence deduces the equations (1), (2), the intermediate steps of the demonstration being conducted by means of infinite series; the necessity of which is avoided by the preceding investigation.

I proceed to investigate certain results relating to these functions, and to the theory of elliptic functions which have been given by Jacobi in two papers, "Suite des notices sur les fonctions elliptiques," *Crelle*, tom. III. p. 306, and tom. IV. p. 185, but without demonstration.

In the first place, the equation

$$\frac{d^2\Sigma}{du^2} + 2u\left(k^2 - \frac{E}{K}\right)\frac{d\Sigma}{du} + 2kk'^2\frac{d\Sigma}{dk} = 0 \; \ldots\ldots \; (4)$$

is satisfied by $\Sigma = \Theta(u)$ or $\Sigma = \mathrm{H}(u)$. It will be sufficient to prove this for $\Sigma = \Theta(u)$, since a similar demonstration may easily be found for the other value. The following preliminary formulæ will be required:

$$k\frac{dK}{dk} = \frac{E}{k'^2} - K, \quad k\frac{dE}{dk} = E - K,$$

$$k\frac{dK'}{dk} = -\frac{E'}{k'^2} + \frac{k^2K}{k'^2}, \quad KK' - EK' - E'K = -\frac{\pi}{2},$$

which are all of them known.

Now, writing $\Theta(u)$ under the slightly more convenient form

$$\Theta u = \sqrt{\left(\frac{2Kk'}{\pi}\right)}\, e^{\int_0 du \int_0 du\, dn^2 u - \frac{1}{2}u^2 \frac{E}{K}},$$

we have $$\frac{d\Theta u}{du} = \left(\int_0 du\, dn^2 u - \frac{E}{K}u\right)\Theta u$$

$$= \left\{u\left(k'^2 - \frac{E}{K}\right) + k^2\int_0 du\, cn^2 u\right\}\Theta u,$$

$$\frac{d^2\Theta u}{du^2} = \left[dn^2 u - \frac{E}{K} + \left\{u\left(k'^2 - \frac{E}{K}\right) + k^2\int_0 du\, cn^2 u\right\}^2\right]\Theta u,$$

$$\frac{d\Theta u}{dk} = \left[\frac{1}{2Kk'}\frac{dKk'}{dk} - \frac{1}{2}u^2\frac{d}{dk}\frac{E}{K} + \int_0 du\int_0 du\frac{d}{dk}dn^2 u\right]\Theta u.$$

The success of the process depends upon a transformation of the double integral

$$\int_0 du \int_0 du \, \frac{d}{dk} \, dn^2 u.$$

To effect this we have

$$\frac{d}{dk} \, dn^2 u = -2k sn u \left(sn u + k \frac{d}{dk} sn u \right);$$

but, by a known formula,

$$k'^2 \frac{d}{dk} sn u = -k cn u \, dn u \int_0 cn^2 u \, du + k cn^2 u \, sn u;$$

whence $sn u + k \dfrac{d}{dk} sn u = \dfrac{1}{k'^2} sn u \, dn^2 u - k^2 cn u \, dn u \int_0 du \, cn^2 u,$

or $\dfrac{d}{dk} dn^2 u = -\dfrac{2k}{k'^2} (sn^2 u \, dn^2 u - k^2 sn u \, cn u \, dn u \int_0 du \, cn^2 u)$

$$= -\frac{2k}{k'^2} \left\{ sn^2 u \, dn^2 u + \tfrac{1}{2} k^2 \left(\frac{d}{du} cn^2 u \right) \int_0 du \, cn^2 u \right\};$$

whence $\int_0 du \int_0 du \, \dfrac{d}{dk} dn^2 u$

$$= -\frac{2k}{k'^2} \left\{ \int_0 du \int_0 du \, sn^2 u \, dn^2 u + \tfrac{1}{2} k^2 \int_0 du \, (cn^2 u \int du \, cn^2 u - \int du \, cn^4 u) \right\}$$

$$= -\frac{k}{k'^2} \left\{ \int_0 du \int_0 du \, (2 sn^2 u \, dn^2 u - k^2 cn^4 u) + \tfrac{1}{2} k^2 \left(\int_0 du \, cn^2 u \right)^2 \right\}.$$

But $\dfrac{d^2}{du^2} sn^2 u = 2 (cn^2 u \, dn^2 u - sn^2 u \, dn^2 u - k^2 sn^2 u \, cn^2 u)$

$$= 2 (k'^2 - 2 sn^2 u \, dn^2 u + k^2 cn^4 u);$$

or, integrating,

$$sn^2 u = k'^2 u^2 - 2 \int_0 du \int_0 du \, (2 sn^2 u \, dn^2 u - k^2 cn^4 u);$$

whence at length

$$\int_0 du \int_0 du \, \frac{d}{dk} dn^2 u = -\tfrac{1}{2} k u^2 + \tfrac{1}{2} \frac{k}{k'^2} sn^2 u + \frac{k^3}{2k'^2} \left(\int_0 du \, cn^2 u \right)^2.$$

Also $\dfrac{d}{dk} Kk' = \dfrac{E - K}{Kk'}, \quad \dfrac{d}{dk} \dfrac{E}{K} = \dfrac{1}{kk'^2} \left\{ k'^2 \left(\dfrac{2E}{K} - 1 \right) - \dfrac{E^2}{K^2} \right\},$

so that

$$\frac{d \Theta u}{dk} = \frac{1}{2kk'^2} \left\{ \frac{E}{K} - dn^2 u - u^2 \left(k'^2 - \frac{E}{K} \right)^2 - k^4 \left(\int du \, cn^2 u \right)^2 \right\} \Theta u.$$

And substituting these values of $\frac{d}{du}\Theta u$, $\frac{d^2}{du^2}\Theta u$ and $\frac{d}{dk}\Theta u$ in the equation (4) in the place of the corresponding differential coefficients of Σ, all the terms vanish, or the equation is satisfied by $\Sigma = \Theta(u)$, and similarly it would be satisfied by $\Sigma = \mathrm{H}(u)$.

Assume now $\quad \omega = \frac{\pi K'}{K}, \quad v = \frac{\pi u}{2K}.$

Then observing the equation

$$\frac{d}{dk}\,\frac{K'}{K} = \frac{1}{K^2 kk'^2}(KK' - KE' - K'E) = -\frac{\pi}{2K^2 kk'^2},$$

we have $\quad \frac{d\Sigma}{du} = \frac{\pi}{2K}\,\frac{d\Sigma}{dv}, \quad \frac{d^2\Sigma}{du^2} = \frac{\pi^2}{4K^2}\,\frac{d^2\Sigma}{dv^2},$

$$\frac{d\Sigma}{dk} = \frac{v}{kk'^2}\left(k'^2 - \frac{E}{K}\right)\frac{d\Sigma}{dv} - \frac{\pi^2}{2K^2 kk'^2}\,\frac{d\Sigma}{d\omega};$$

whence, substituting in the equation (4), this becomes

$$\frac{d^2\Sigma}{dv^2} - 4\,\frac{d\Sigma}{d\omega} = 0 \;\ldots\ldots\ldots\ldots\; (5),$$

which is of course satisfied as before by $\Sigma = \Theta(u)$, or $\Sigma = \mathrm{H}(u)$, an equation demonstrated in a different manner (by means of expansions) by Jacobi in the Memoirs quoted.

Consider next the equation

$$\frac{d^2\Sigma}{du^2} - 2nu\left(k'^2 - \frac{E}{K}\right)\frac{d\Sigma}{du} + 2nkk'^2\,\frac{d\Sigma}{dk} = 0 \;\ldots\; (6),$$

(n being any positive integer number). Then, by assuming

$$\omega = n\,\frac{\pi K'}{K}, \quad v = \frac{n\pi u}{K},$$

we should be led as before to the equation (5). Hence, considering Θu or $\mathrm{H}u$ as functions of u and $\frac{K'}{K}$, the equation (6) is satisfied by assuming for Σ a corresponding function of nu and $\frac{nK'}{K}$. Let λ be the modulus corresponding to a transformation of the n^{th} order; then Λ, Λ' being the complete functions corresponding to this modulus,

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$\dfrac{\Lambda'}{\Lambda} = n\,\dfrac{K'}{K}$, so that the equation (6) will be satisfied by assuming $\Sigma = \Theta_{,}(nu)$ or $\Sigma = H_{,}(nu)$, where $\Theta_{,}$, $H_{,}$ correspond to the new modulus λ.

Assume now in the equation (6),

$$\Sigma = \left(\frac{\pi}{2}\right)^{\frac{n-1}{2}} (KK')^{-\frac{n-1}{2}}\, \Theta^n u . z.$$

Hence, substituting,

$$\frac{d^2}{du^2}(\Theta^n u . z) - 2nu\left(k'^2 - \frac{E}{K}\right)\frac{d}{du}(\Theta^n u . z)$$

$$+ 2nkk'^2 (KK')^{\frac{n-1}{2}} \frac{d}{dk}\left[(KK')^{-\frac{n-1}{2}}\Theta^n u . z\right] = 0:$$

but $(KK')^{\frac{n-1}{2}} \dfrac{d}{dk}\left[(KK')^{-\frac{n-1}{2}}\Theta^n u . z\right] = \dfrac{d}{dk}(\Theta^n u . z) - \dfrac{n-1}{2KK'}\,\dfrac{dKK'}{dk}\,\Theta^n u . z$,

or effecting the differentiation, and eliminating $\dfrac{d\Theta u}{dk}$ by means of the equation obtained from (4) by writing $\Sigma = \Theta u$,

$$(KK')^{\frac{n-1}{2}} \frac{d}{dk}\left[(KK')^{-\frac{n-1}{2}}\Theta^n u . z\right]$$

$$= \Theta^n u\left[\frac{dz}{dk} - \frac{nz}{2kk'^2\Theta u}\left\{\frac{d^2\Theta u}{du^2} - 2\left(k'^2 - \frac{E}{K}\right)\frac{d\Theta u}{du}\right\} + \frac{n-1}{2kk'^2}\left(1 - \frac{E}{K}\right)z\right].$$

Substituting in (6) and reducing,

$$\frac{d^2 z}{du^2} + 2n\left[\frac{1}{\Theta n}\,\frac{d\Theta u}{du} - u\left(k'^2 - \frac{E}{K}\right)\right]\frac{dz}{du} + 2nkk'^2\frac{dz}{dk}$$

$$+ n(n-1)\left\{\left[\frac{1}{\Theta^2 u}\left(\frac{d\Theta u}{du}\right)^2 - \frac{1}{\Theta u}\,\frac{d^2\Theta u}{du^2}\right] + \left(1 - \frac{E}{K}\right)\right\} z = 0,$$

i.e. $$\frac{d^2 z}{du^2} + 2n\left[\frac{d\log\Theta u}{du} - u\left(k'^2 - \frac{E}{K}\right)\right]\frac{dz}{du} + 2nkk'^2\frac{dz}{dk}$$

$$+ n(n-1)\left[-\frac{d^2\log\Theta u}{du^2} + \left(1 - \frac{E}{K}\right)\right] z = 0$$

But $$\frac{d\log\Theta u}{du} = u\left(k'^2 - \frac{E}{K}\right) + k^2 \int_0 du\, cn^2 u,$$

$$\frac{d^2\log\Theta u}{du^2} = 1 - \frac{E}{K} - k^2 sn^2 u;$$

$$n(n-1)x^2z + (n-1)(ax - 2x^3)\frac{dz}{dx}$$

$$+ (1 - ax^2 + x^4)\frac{d^2z}{dx^2} - 2n(a^2 - 4)\frac{dz}{da} = 0 \;\ldots\; (8);$$

which is therefore satisfied by assuming for z either the numerator or the denominator of $\sqrt{\lambda}\, sn_1 u$ (the transformation of the n^{th} order), which is the form in which the property is given by Jacobi.

In the case where n is odd, the denominator is of the form

$$B_0 + B_1x^2 \ldots . + B_{\frac{1}{2}(n-1)}\, x^{n-1},$$

and then the numerator is

$$x\left(B_{\frac{1}{2}(n-1)} \ldots . + B_1x^{n-3} + B_0x^{n-1}\right),$$

where $B_0 = \sqrt{\left(\frac{\lambda'}{k'M}\right)}, \quad B_{\frac{1}{2}(n-1)} = \sqrt{\left(\frac{\lambda\lambda'}{kk'M^2}\right)};$

and all the remaining coefficients may be determined from these, the modular equation being supposed known. But the principal use of the formula is for the multiplication of elliptic functions, which it is well known corresponds to the case where n is a square number. Writing $n = \nu^2$, when ν is odd, the denominator is

$$1 + B_2x^4 \ldots . + B_{\frac{1}{2}(\nu^2-3)}\, x^{\nu^2-3} \pm \nu x^{\nu^2-1},$$

(the $\pm$ sign according as $\nu = (4p+1)$ or $(4p-1)$); and the

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numerator is obtained from this by multiplying by x and reversing the order of the coefficients. When ν is even the denominator is

$$1 + B_2 x^4 \dots \pm B_2 x^{\nu^2-4} \pm x^{\nu^2},$$

(+ or −, according as $\nu = 4p$ or $\nu = 4p + 2$), so that there are only half as many coefficients to be determined; but then the numerator must be separately investigated. In general, by leaving (n) indeterminate, and integrating in the form of a series arranged according to ascending powers of x^2; then, whenever n is a square number, the series terminates and gives the denominator of the corresponding formula of multiplication; but the general form of the coefficients has not hitherto been discovered.

By writing $\frac{x}{\sqrt{n}}$ instead of x, and then making n infinite, the equation (8) takes the form

$$x^2 z + ax\frac{dz}{dx} + \frac{d^2z}{dx^2} - 2(a^2-4)\frac{dz}{da} = 0 \dots (9):$$

and it is worth while, before attempting the solution of the general case, to discuss this more simple one.*

Assume $z = 1 + C_1 \frac{x^2}{1.2} \dots + C_r \frac{x^{2r}}{1.2 \dots 2r} + \dots;$

then it is easy to obtain

$$C_{r+2} = -(2r+1)(2r+2)C_r - (2r+2)aC_{r+1} + 2(a^2-4)\frac{dC_{r+1}}{da}.$$

The general form may be seen to be

$$C_r = (-)^{r+1}\left\{2^{2r-3}C_r^1 a^{r-3} + 2^{2r-6}C_r^2 a^{r-4} + \dots\right\},$$

and then

$$C^p_{r+1} - pC_r^p = -r(2r-1)C_{r-1}^{p-1} + 16(r+2-2p)C_r^{p-1}.$$

The complete value of C_r^p (assuming $C_r^0 = 0$) is given by an equation of the form

$$C_r^p = {}^0C_r^p + {}^1C_r^p 2^r + {}^2C_r^p 3^r \dots + {}^{p-1}C_r^p p^r,$$

---

* Writing $(\beta+2)$ for a, and putting $z = e^{+\frac{1}{4}x^2}\rho$, this becomes

$$\frac{d^2\rho}{dx^2} - \rho = \beta x^2\rho - \beta x\frac{d\rho}{dx} + (8\beta + 2\beta^2)\frac{d\rho}{d\beta};$$

and if $\rho = \Sigma Z_n\beta^n$,

$$\frac{d^2 Z_n}{dx^2} - (8n+1)Z_n = \left(x^2 + 2n - 2 - x\frac{d}{dx}\right)Z_{n-1};$$

from which the successive values of Z_0, Z_1, &c. might be calculated.

where ${}^{0}C_r^p$, ¹
degrees $2p$ –
completely to effect the integration, and my only object is to give an idea of the law of the successive terms, it will be sufficient to consider the first or algebraical term ${}^{0}C_r^p$, which is determined by the same equation as C_r^p, and moreover completely determined by this equation and the single additional relation $C_r^1 = 1$, since the arbitrary constants of the integration affect only the terms multiplied by 2^r, 3^r, &c.

Assume C_r^p

$$=\frac{1}{[p-1]^{p-1}}\left\{2^{r-1}L^p[r-2]^{2p-2}+3^{r-3}M^p[r-3]^{2p-3}+\dots 2^{r+1}X^p[r-2p]^p\right\};$$

and substituting this value,

$$\begin{aligned}
&(1-p)\,L^p &&=(1-p)\left\{L^{p-1}\right\},\\
&(1-p)\,M^p-2p(2-2p)\,L^p &&=(1-p)\left\{M^{p-1}-11L^{p-1}\right\},\\
&(1-p)\,N^p-2p(3-2p)\,M^p &&=(1-p)\left\{N^{p-1}-\ 7M^{p-1}+12L^{p-1}\right\},\\
&(1-p)\,O^p-2p(4-2p)\,N^p &&=(1-p)\left\{O^{p-1}-\ 3N^{p-1}+30M^{p-1}\right\},\\
&\quad\vdots &&\qquad\quad\vdots
\end{aligned}$$

the law of which is obvious, the coefficients on the second side in the qth line being 1, $4q-19$, and $(2q-3)(2q-2)$ respectively. By successive integrations and substitutions

$$\begin{aligned}
&L^p - L^{p-1} = 0, && L^p = 1,\\
&M^p - M^{p-1} = 4p - 11, && M^p = (p-1)(2p-7),\\
&N^p - N^{p-1} = -8p^3 + 26p^2 + 49p - 114\,; && \vdots\\
&\vdots
\end{aligned}$$

(the constants determined by $M^1 = 0$, $N^1 = 0$, $O^2 = 0$, $P^2 = 0, \dots$ so as to make C_r^p contain positive powers only of r).

The following are a few of the complete values of C_r^p, the constants determined so as to satisfy $C^p_{p+1} = 0$ (except $C_2^1 = 1$), and the factorials being partially developed in powers of r, viz.

$$\begin{aligned}
&C_r^1 = 1,\\
&C_r^2 = (r-3)(2r-7),\\
&C_r^3 = \tfrac{1}{2}(r-4)(r-5)(4r^2-24r+51),\\
&C_r^4 = \tfrac{1}{6}\left\{(r+5)(r-6)(r-7)(8r^3-60r^2+286r+63)\right.\\
&\qquad\quad \left.+\,384\,(9r^2-93.r+242-2.4^r)\right\},\\
&\text{\&c.}
\end{aligned}$$

(it is curious that C_5^4, C_6^4, C_7^4, all three of them vanish). It seems hopeless to continue this investigation any further.

Returning to the equation (8), and assuming for z an expression of the same form as before, we have, corresponding to the equations before found for the coefficients C_r,

$$C_{r+2} = -(2r+1)(2r+2)(n-2r)(n-2r-1)\,C_r$$
$$-(2r+2)(n-2r-2)\,aC_{r+1} + 2n(a^2-4)\frac{dC_{r+1}}{da}.$$

The case corresponding to the denominator in the multiplication of elliptic functions is that of $C_0 = 1$, $C_1 = 0$. It is easy to form the table—

$$C_0 = 1,$$
$$C_1 = 0,$$
$$C_2 = -2\,n(n-1),$$
$$C_3 = \quad 8\,n(n-1)(n-4)\,a,$$
$$C_4 = -4\,n(n-1)(n-4)\,[n+75] - 32n(n-1)(n-4)(n-9)\,a^2,$$
$$C_5 = \quad 96\,n(n-1)(n-4)(n-9)\,[n+44]\,a$$
$$+128\,n(n-1)(n-4)(n-9)(n-16)\,a^3,$$
$$C_6 = -24\,n(n-1)(n-4)(n-9)\,[17n^2+403n+9000]$$
$$-960\,n(n-1)(n-4)(n-9)(n-16)\,[n+41]\,a^2$$
$$-512\,n(n-1)(n-4)(n-9)(n-16)(n-25)\,a^4,$$
$$C_7 = \quad 96\,n(n-1)(n-4)(n-9)(n-16)\,[79n^2+2825n+36180]\,a$$
$$+7168\,n(n-1)(n-4)(n-9)(n-16)(n-25)\,[n+42]\,a^3$$
$$+2048\,n(n-1)(n-4)(n-9)(n-16)(n-25)(n-36)\,a^5,$$
$$C_8 = \quad 48\,n(n-1)(n-4)(n-9)\,[283n^4 - 26978n^3 + 277827n^2$$
$$-5491932n + 127764000]$$
$$-3840\,n(n-1)(n-4)(n-9)(n-16)(n-25)\times$$
$$[23n^2+1069n+23436]\,a^2$$
$$-15360\,n(n-1)(n-4)(n-9)(n-16)(n-25)(n-36)\times$$
$$[3n+133]\,a^4$$
$$-8192\,n(n-1)(n-4)(n-9)(n-16)(n-25)(n-36)(n-49)\,a^6,$$

&c.

in which of course the coefficient of the highest power of n, in the successive coefficients C_r, is the value of C_r obtained from the equation (8). With regard to the law of these coefficients I have found that

$$C_r = (-)^{r+1}\,2^{2r-3}\,n(n-1^2)\ldots\{n-(r-1)^2\}\,C_r^{\,1}a^{r-2}$$
$$+\,2^{2r-6}\,n(n-1^2)\ldots\{n-(r-2)^2\}\,C_r^{\,2}a^{r-4}$$
$$+\,2^{2r-9}\,n(n-1^2)\ldots\{n-(r-3)^2\}\,C_r^{\,3}a^{r-6}$$
$$+\,\&c.$$

(where however the next term does not contain, as would at first sight be supposed, the factor $n(n-1^2)\ldots\{n-(r-4)^2\}$.) And then

$$C^1 = 1,$$
$$C^2 = (r-3)\left[n(2r-7)+(r-1)(3r-7)\right],$$
$$C_r^3 = \tfrac{1}{2}(r-4)(r-5)\left[n^2(4r^2-24r+51) + n(32r^3-220r^2+412r-255) + 2(r-1)(r-2)(32r^2-88r+51)\right].$$

In conclusion may be given the following results, in which, recapitulating the notation

$$x = \sqrt{k}\,\mathrm{sn}\,u, \quad a = k + \frac{1}{k}, \quad \Delta x = \sqrt{(1-ax^2+x^4)},$$

$$\sqrt{k}\,\mathrm{sn}\,2u = \frac{2x\,\Delta x}{1-x^4},$$

$$\sqrt{k}\,\mathrm{sn}\,3u = \frac{x(3-4ax^2+6x^4-x^8)}{1-6x^4+4ax^6-3x^8},$$

$$\sqrt{k}\,\mathrm{sn}\,4u = \frac{4x\,\Delta x\,(1-x^4)(1-2ax^2+6x^4-2ax^6+x^8)}{1-20x^4+32ax^6-(26+16a^2)x^8+32ax^{10}-20x^{12}+x^{16}},$$

$$\sqrt{k}\,\mathrm{sn}\,5u =$$
$$\frac{x\{5-20ax^2+(62+16a^2)x^4-80ax^6-105x^8+360ax^{10}-(300+240a^2)x^{12}+(368a+64a^3)x^{14}-(125+160a^2)x^{16}+140ax^{18}-50x^{20}+x^{24}\}}{\{1-50x^4+140ax^6-(125+160a^2)x^8+(368a+64a^3)x^{10}-(300+240^2)x^{12}+360ax^{14}-105x^{16}-80ax^{18}+(62+16a^2)x^{20}-20ax^{22}+5x^{24}\}}$$

&c.

Thus, writing $-x^2$ for x^2, $k=1$, and $\therefore a=2$,

$$\tan 3u = x\,\frac{(3+8x^2+6x^4-x^6)}{1-6x^4-8x^6-3x^8} = \frac{x(3-x^2)(1+x^2)^3}{(1-3x^2)(1+x^2)^3} = \frac{x(3-x^2)}{1-3x^2},$$

where $x = \tan u$. (And in general in reducing $\tan nu$ the extraneous factor in the numerator and denominator is $(1+x^2)^{\frac{1}{2}n(n-1)}$.)

58, Chancery Lane, London, May 17, 1847.

(To be continued.)

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ON CERTAIN ALGEBRAIC FUNCTIONS.

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I. A HOMOGENEOUS function of the second degree and of m undetermined quantities ξ', ξ'',., $\xi^{(m)}$, may be written as follows:—

$$\kappa_1'^2\xi'^2 + 2(\kappa_1''\xi'' + \kappa_1'''\xi''' + . + \kappa_1^{(m)}\xi^{(m)})\kappa_1'\xi' + f(\xi'', \xi''', .., \xi^{(m)}) ...(1);$$

add to, and subtract from, this expression the square of half the coefficient of $\kappa_1'\xi'$, and let

$$\kappa_1'\xi' + \kappa_1''\xi'' + . + \kappa_1^{(m)}\xi^{(m)} = h_1:$$

then (1) may be put under the form

$$h_1^2 + \phi(\xi'', \xi''', .., \xi^{(m)}).$$

In like manner $\phi(\xi'', \xi''', .., \xi^{(m)})$, which is a homogeneous function of the second degree and of $m-1$ undetermined quantities ξ'', ξ''', .., $\xi^{(m)}$, may be written thus:—

$$\kappa_2''^2\xi''^2 + 2(\kappa_2'''\xi''' + \kappa_2^{iv}\xi^{iv} + . + \kappa_2^{(m)}\xi^{(m)})\kappa_2''\xi'' + \chi(\xi''', \xi^{iv}, .., \xi^{(m)}),$$

which expression may, by proceeding as before, be reduced to the form

$$h_2^2 + \psi(\xi''', \xi^{iv}, .., \xi^{(m)}),$$

where $$h_2 = \kappa_2''\xi'' + \kappa_2'''\xi''' + . + \kappa_2^{(m)}\xi^{(m)};$$

hence (1) may be represented by

$$h_1^2 + h_2^2 + \psi(\xi''', \xi^{iv}, .., \xi^{(m)});$$

and if we reduce ψ and the corresponding subsequent functions, as we have already done ϕ and the given one, we may put (1) under the form

$$h_1^2 + h_2^2 + . + h_r^2 + . + h_m^2,$$

where $$h_r = \kappa_r^{(r)}\xi^{(r)} + \kappa_r^{(r+1)}\xi^{(r+1)} + .. \kappa_r^{(m)}\xi^{(m)}.$$

II. A homogeneous function of the third degree and of v undetermined quantities Ξ', ξ', .., $\xi^{(v-1)}$ may be written thus:

$$K'^3\Xi'^3 + 3A'K'^2\Xi'^2 + B'K'\Xi' + C' \ldots\ldots (2),$$

where A', B', and C' are free from Ξ; and this expression again may be put under the form

$$(K'\Xi' + A')^3 + (B' - 3A'^2)K'\Xi' + C' - A'^3 \ldots (3).$$

Let $$K'\Xi' + A' = \mathrm{h}_1;$$

then, since $B' - 3A'^2$ is a homogeneous function of the second

degree and of $v-1$ undetermined quantities $\xi', \xi'', .., \xi^{(v-1)}$, we may (by the processes of paragraph I.) put (3) under the form

$$h_1^3 + (k_1^2 + k_2^2 + . + k_{v-1}^2)\,K'\Xi' + C' - A''\ldots\ldots(4);$$

now, the $v-1$ quantities $\xi', \xi'', .., \xi^{(v-1)}$, being perfectly undetermined, we may make

$$k_1^2 + k_2^2 = 0, \quad k_3^2 + k_4^2 = 0, .., .$$

or $$k_1 + (1)^{\frac{1}{2}} k_2 = 0, \quad k_3 + (1)^{\frac{1}{2}} k_4 = 0, .., \ldots\ldots(5);$$

the last of which lower line of equations is, ($v-1$ being supposed even),

$$k_{v-2}^2 + k_{v-1}^2 = 0.$$

By means of the system (5) of linear equations, eliminate, from $C' - A''$, (which is free from Ξ) $v-1$ of the ξ's, and let $f^a(b)$ denote a homogeneous function of the a^{th} degree and of b undetermined quantities; then, since the coefficient of $K'\Xi'$ has been made to vanish, (4) may now be represented by

$$h_1^3 + f^3\left(\frac{v-1}{2}\right),$$

for v write v_1 and let $v_{x+1} = \dfrac{v_x - 1}{2}$; then, by processes similar to those which we have just employed, we may reduce the given function of the third degree, to the form

$$h_1^3 + h_2^3 + . + h_x^3 + f^3(v_{x+1}).$$

The equation of finite differences which gives v_x is

$$v_{x+1} - \tfrac{1}{2}\,v_x = -\tfrac{1}{2},$$

of which the solution is

$$v_x = -1 + C(\tfrac{1}{2})^{x-1}\ldots\ldots\ldots\ldots\ldots(6).$$

Now in order that the given function of the third degree may be reduced to the form of a sum of m cubes, and may, after such reduction, still involve m undetermined quantities, the cubic

$$f^3(v_{m+1}) = 0,$$

to which we shall be conducted when we have arrived at h_m^3, ought to contain *two* undetermined quantities; more than two would be superfluous, hence

$$v_{m+1} = 2 = -1 + C(\tfrac{1}{2})^m \text{ (by 6)},$$

therefore $C = 3.2^m$, and v_1 (or v) $= 3.2^m - 1$;

also $v_x = 3.2^{m-x+1} - 1$;

and $v_x - 1$ is, of course, even.

which, by proceeding in the same manner, may be further reduced to

$$f^5(w_m) = \mathfrak{h}_1^5 + \mathfrak{h}_2^5 + . + \mathfrak{h}_n^5 + f^5(w_{m-n});$$

and reasoning in the same manner as in the last paragraph, we infer that the equation (10), combined with the following,

$$w_{m-m} = w_0 = 2,$$

will completely determine w_m. This investigation differs from the preceding ones as follows,—the others give absolute results, but this last leaves us an equation of the fifth degree to solve; so that all that we can say is, that we have made the difficulty of reducing $f^5(u_m)$ to the form

$$\mathfrak{h}_1^5 + \mathfrak{h}_2^5 + . + \mathfrak{h}_m^5$$

(where $\mathfrak{h}_r$ involves $m - r + 1$ undetermined quantities) depend upon that of solving an equation of the fifth degree. The discussion of the equations of differences above given as well as of that which occurs in the succeeding paragraph must be deferred till another opportunity.

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V. For u write ${}_4u$, and for w write ${}_5u$, then if ${}_ru_m$ denote the number of disposable quantities necessary in order that an algebraic function of the r^{th} degree of those quantities may be made to satisfy the condition

$$f^r({}_ru_m) = {}^{\backprime}h_1^{\,r} + {}^{\backprime}h_2^{\,r} + . + {}^{\backprime}h_m^{\,r}$$

(${}^{\backprime}h$ having the same meaning as H, h, h, $\mathfrak{h}$, &c. and ${}^{\backprime}h$, involving $m - s + 1$ undetermined quantities), it will be found that the equation of finite differences

$${}_ru_{s+1} - 3.2^{1 + 2^{{}_4u^{\,2^{{}_5u^{\cdot^{\cdot^{2^{{}_ru_s}}}}}}}} + 1 = 0,$$

and the subordinate ones implicitly included in it, combined with ${}_ru_0 = 2$, suffice for the determination of ${}_ru_m$, subject to the solution of an equation of the r^{th} degree. This will be seen if we reflect on the preceding paragraphs.

VI. It is hardly necessary to observe that, if m be even, then whenever we can reduce the left hand side of an algebraic equation, of which the right hand side is zero, to the above form, we may, by grouping the h^r's in pairs, equating the sum of each pair to zero, and making an obvious depression of degree, eliminate $\frac{1}{2}m$ of the quantities ξ, &c. between this equation and another, without introducing any elevation of degree by elimination, and without having to solve any equation of a degree higher than the higher of the two given equations.

VII. I now proceed to inquire what is the number of disposable quantities requisite in order that we may, by means of equations whose degrees shall not exceed the r^{th}, simultaneously satisfy a equations of the r^{th} degree, b of the $(r - 1)^{\text{th}}$, c of the $(r - 2)^{\text{th}}$, . , β of the second, and α of the first degree between those quantities.

VIII. Call the equations of the r^{th} degree the 1^{st}, 2^{d}, and a^{th}, respectively; then, if we can reduce the $(a - 1)^{\text{th}}$ equation to the form

$$h_1^{\,r} + h_2^{\,r} = 0,$$

the a^{th} equation will be solvable without elevation of degree arising from elimination. Now it will be seen that the processes employed in the reductions here treated of do not in any case conduct to an equation of a degree higher than the r^{th}. So that if we had

$${}_ru_2 \;.$$

disposable quantities, we might reduce the solution of the

$(s-1)^{th}$ and s^{th} equations to that of two equations of the r^{th}, and others of lower degrees.

IX. Again, in order that we may avoid elevation of degree from the $(s-2)^{th}$ equation, we must reduce it to the sum of $2_r u_s$ powers, and then group the powers two and two and eliminate. But this requires that we should have

$${}_r u_{2_r u_s}$$

disposable quantities, and the whole of the equations of the r^{th} degree will require that we should have

$${}_r u_{2_r u_{2_r u_{\cdot_{\cdot_{2_r u_s}}}}}$$

disposable quantities; in which expression fully written the letter u would occur $s-1$ times. Call this expression U or $u\left[{r \atop s}\right]$. Then, from considerations similar to those which have conducted us to the above expression, we find that the b equations of the $(r-1)^{th}$ degree will increase this last expression to

$${}_{r-1} u_{2_{r-1} u_{2_{r-1} u_{\cdot_{\cdot_{2_{r-1} u_U}}}}};$$

in which expression, when fully written, u would occur b times. Call this last expression

$$u\begin{bmatrix} r-1, & r \\ b, & a \end{bmatrix};$$

then these operations must be carried on till we have ascertained how many quantities are required for the reduction of all the equations from the fourth degree upwards. The expression for the number of quantities requisite for this purpose will, on the principle of the above notation, be represented by

$$u\begin{bmatrix} 4, & 5, \ldots, & r-1, & r \\ \delta, & \varepsilon, \ldots, & b, & a \end{bmatrix}.$$

Then, if the equations of the second and third degrees be treated by obvious extensions of processes which I have above given and if the equations of the first degree be also taken into consideration, we shall find that the formula which expresses the number of arbitrary quantities necessary in order that we may make the solution of a equations of

the r^{th} degree, b of the $(r-1)^{\text{th}}$, ., γ of the third, β of the second, and a of the first, between the same unknowns depend upon the equations of the r^{th}, $(r-1)^{\text{th}}$, and lower degrees only, is

$$a + 2^{\beta}\left(3.2^{3.2^{3.2^{3.2^{\cdot^{\cdot^{\cdot^{3.2^{1+2u\begin{bmatrix}4, & 5, & ., & r-1, & r\\ \delta, & \epsilon, & ., & b, & a\end{bmatrix}}}}}} - 1} - 1} - 2} - 1\right),$$

$$\text{or } \Upsilon;$$

in which formula there are supposed to be γ lines of exponents, γ being the number of equations of the third degree which it is required to satisfy. Were these formulæ incapable of giving illusory results, there would remain but little to be done in the Theory of Algebraic Equations; the formulæ would also have other applications. But in order to ascertain the limits of their application, and to compare particular cases of them with corresponding cases in Mr. Jerrard's method, we must ascertain the results which follow from supposing $\Upsilon - v$ of the quantities ξ, ξ', &c. to be functions of the remaining v of them. This point I shall defer, but I hope to discuss the question at some future time in its most general form and so as to include the isolated results at which I have already arrived in a contemporary periodical.*

This paper contains, I think, everything necessary for the complete development of the method of which I have already given isolated discussions in the *Philosophical Magazine*, and the *Mathematician*.

Postscript. June 14, 1847. The reduction which I suggested in a short note, published at pages 285–286 of vol. I. of the present series of this work, will be found to simplify my discussion at page 105 of vol. III. of the former series.

2, *Church-Yard Court, Temple,*
February 25, 1847.

* The *London, Edinburgh, and Dublin Philosophical Magazine.*

$$x_1 y_1 + x_2 y_2 - x_3 y_3 = 0 \ldots\ldots\ldots\ldots (16),$$

$$x_1 z_1 + x_2 z_2 - x_3 z_3 = 0 \ldots\ldots\ldots\ldots (17),$$

$$y_1 z_1 + y_2 z_2 - y_3 z_3 = 0 \ldots\ldots\ldots\ldots (18).$$

$$\pm\frac{x_1}{a} = \frac{y_2 z_3 - y_3 z_2}{bc},\ \pm\frac{x_2}{a} = \frac{y_3 z_1 - y_1 z_3}{bc},\ \mp\frac{x_3}{a} = \frac{y_1 z_2 - y_2 z_1}{bc} \ldots (19),$$

$$\pm\frac{y_1}{b} = \frac{x_3 z_2 - x_2 z_3}{ac},\ \pm\frac{y_2}{b} = \frac{x_1 z_3 - x_3 z_1}{ac},\ \mp\frac{y_3}{b} = \frac{x_2 z_1 - x_1 z_2}{ac} \ldots (20),$$

$$\mp\frac{z_1}{c} = \frac{x_2 y_3 - x_3 y_2}{ab},\ \mp\frac{z_2}{c} = \frac{x_3 y_1 - x_1 y_3}{ab},\ \pm\frac{z_3}{c} = \frac{x_1 y_2 - x_2 y_1}{ab} \ldots (21).$$

The preceding equations are analogous to those given in my paper entitled "Investigation of certain Properties of the Ellipsoid," (see *Journal*, New Series, vol. II., pp. 13–19), of which communication this is designed to form a continuation. Most of the theorems deduced in that paper are true of conjugate hyperboloids with some slight modifications, the chief of which arise from the squares of all the lines that refer to the hyperboloid of two sheets (those lines which have 3 subscribed) having a negative sign.

As the investigations of the following properties of conjugate hyperboloids, (a)....(g), are the same as those of the analogous ones (A)....(G), in the paper just mentioned, I shall, for brevity's sake, omit them; also since the enunciations of (b) and (c) would be the same as those of (B) and (C), the latter will, for the same reason, be simply referred to.

(a) If three conjugate points be projected on any diametral plane by lines drawn parallel to the diameter conjugate to this plane, the difference between the square of the line of projection drawn from the point on the hyperboloid of two sheets, and the sum of the squares of the other two lines, is equal to the square of the semidiameter.

(b) The same as (B), see *Journal*, vol. II. p. 15, New Series.

(c) The same as (C), *ib.*

(d) The difference between the square of any diameter of the hyperboloid of two sheets, and the sum of the squares of two conjugate diameters (which appertain to the hyperboloid of one sheet) is constant. Hence, should it happen that $a^2 + b^2 = c^2$, the square of any diameter of the hyperboloid of two sheets will be equal to the sum of the squares of two conjugate diameters.

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(e) Each conjugate parallelepiped* is equal to that constructed on the principal diameters.

(f) The difference between the sum of the squares of those two faces of any conjugate parallelepiped which touch the hyperboloid of two sheets, and the sum of the squares of the other four faces (which touch the hyperboloid of one sheet) is constant. Hence, if $a^2b^2 = a^2c^2 + b^2c^2$ or $\frac{1}{c^2} = \frac{1}{a^2} + \frac{1}{b^2}$, the sum of the squares of the two faces will be equal to the sum of the squares of the four.

(g) If perpendiculars be drawn from the centre of conjugate hyperboloids on any three conjugate tangent planes, the difference between the square of the reciprocal of the perpendicular on the tangent plane to the hyperboloid of two sheets, and the sum of the squares of the reciprocals of the other two perpendiculars, is constant. Hence if $\frac{1}{c^2} = \frac{1}{a^2} + \frac{1}{b^2}$, the square of the reciprocal of the former perpendicular will be equal to the sum of the squares of the reciprocals of the other two.

The locus of the intersections of three conjugate tangent planes will be obtained by eliminating x_1, x_2, &c. from (4, 5, 6), and this elimination is at once effected by deducting the square of (6) from the sum of the squares of (4, 5), and reducing by (13)....(18); therefore

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Hence

(h) The locus of the intersections of conjugate tangent planes to conjugate hyperboloids is the hyperboloid of one sheet itself. Hence, also,

(i) Every conjugate parallelepiped to conjugate hyperboloids is inscribed in the hyperboloid of one sheet itself.

The theorems (h, i) which differ widely from the corresponding propositions (H, I) for the ellipsoid, are very remarkable, and, I must confess, these results were totally unexpected by me. Recollecting that conjugate parallelepipeds to conjugate hyperboloids are in some respects analogous to conjugate parallelograms to conjugate hyperbolas,

* The faces of a *conjugate parallelepiped* touch the conjugate hyperboloids and are parallel to conjugate diametral planes.

I imagined that the locus would be the asymptotic cone (3). It is not, however, conjugate parallelepipeds, but conjugate cylinders (to be noticed presently) that are here analogous to conjugate parallelograms.

Theorems in reference to conjugate hyperboloids have now been given analogous to all those contained in the paper on the ellipsoid, except to (K), (L), and (M), and I have not been able to discover any properties similar to these. (In consequence none of the following propositions will be marked (k), (l), or (m).) I shall now introduce a few additional properties of the hyperboloids, several of which have analogues in Plane Geometry.

Referring the conjugate hyperboloids (1, 2) and the asymptotic cone (3) to any conjugate diameters, $A_1'OA_1 = 2a_1$, $B_1'OB_1 = 2b_1$, and $C_1'OC_1 = 2c_1$, their equations will be

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - \frac{z^2}{c_1^2} = 1 \ldots\ldots\ldots\ldots\ldots (22),$$

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - \frac{z^2}{c_1^2} = -1 \ldots\ldots\ldots\ldots (23),$$

and

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - \frac{z^2}{c_1^2} = 0 \ldots\ldots\ldots\ldots\ldots (24).$$

The equations of the tangent planes at C_1' and C_1 to the hyperboloid of two sheets (23), are $z = -c_1$, and $z = c_1$; and either of these values of z substituted in (24) gives $\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1$, for the equation to the section of the cone by the tangent plane at C_1' or C_1; now this is the very same equation we should get by putting $z = 0$ in (22); hence

(n) If sections of the asymptotic cone be made by two parallel tangent planes to the hyperboloid of two sheets, each section (which is an ellipse) is equal, similar and similarly posited, to the section of the conjugate hyperboloid made by a parallel diametral plane.

Again, let sections of (22, 23, 24) be made by the plane $z = z_1$, parallel to that of xy. The areas of these sections being denoted by L, M, N, respectively, and the angle A_1OB_1 by ϕ, we shall evidently have

$$\mathrm{L} = \pi a_1 b_1 \left(\frac{z_1^2}{c_1^2} + 1\right) \sin\phi, \qquad \mathrm{M} = \pi a_1 b_1 \left(\frac{z_1^2}{c_1^2} - 1\right) \sin\phi,$$

$$\mathrm{N} = \pi a_1 b_1 \frac{z_1^2}{c_1^2} \sin\phi;$$

therefore $L - N = N - M = \pi a_1 b_1 \sin \phi$ = area of parallel diametral section. It is hence easily seen that

(o) If elliptic sections of two conjugate hyperboloids and the asymptotic cone be made by any plane, the area of each of the two elliptic rings bounded by the curves of section will be equal to that of the parallel diametral section, or, (n), to that of the section of the cone made by a parallel tangent plane. Hence also the elliptic rings are equal in area for all parallel sections.

DEF. *A conjugate cylinder* to conjugate hyperboloids has its generators parallel to a diameter of the hyperboloid of two sheets and tangent to the hyperboloid of one sheet, and it is limited by the tangent planes touching at the extremities of the diameter.

It is plain that

(p) The diametral section parallel to the ends of a conjugate cylinder is the locus of the points of contact of the generators with the hyperboloid of one sheet, and hence also, (n), the asymptotic cone is the locus of the perimeters of the ends of all conjugate cylinders.

It is evident that a conjugate cylinder and the corresponding conjugate parallelepiped have the same altitude, and that their bases are in the proportion $\pi : 4$; hence, (e),

(q) All conjugate cylinders are equal to each other, and the volume of each is $2\pi abc$; a, b, c being the principal semidiameters.

Moreover, the portion of the asymptotic cone cut off by a tangent plane to the hyperboloid of two sheets has evidently the same base as the corresponding conjugate cylinder, but only half its altitude; the volume of the former solid is consequently equal to one-sixth of that of the latter. Hence the following remarkable theorem.

(r) The volume of the portion of the asymptotic cone cut off by a tangent plane to an hyperboloid of two sheets is constant and equal to $\frac{1}{3}\pi abc$.

I shall next establish the following theorem.

(s) If from any point in an hyperboloid of two sheets as vertex, a cone be described having its generators parallel to those of the asymptotic cone, the volume of the solid included between the surfaces of these two cones is constant and equal to $\frac{1}{12}\pi abc$.

For the tangent plane at the point will, (r), cut off from the asymptotic cone a solid $U = \frac{1}{3}\pi abc$, and the solid V mentioned in (s) is evidently composed of two solids W similar to U, but of only half the (linear) dimensions; hence $W = \frac{1}{8}U = \frac{1}{24}\pi abc$, and $V = 2W = \frac{1}{12}\pi abc$.

(t) If any straight line be drawn cutting an hyperboloid in U_1, V_1, the conjugate hyperboloid in U_2, V_2, and the asymptotic cone in U, V; then will $UU_1 = VV_1$, $UU_2 = VV_2$, and $UU_1 . U_1V = UV_1 . V_1V = UU_2 . U_2V = UV_2 . V_2V =$ square of the parallel semidiameter.

The truth of (t) may easily be shewn by referring the surfaces to conjugate diameters, one of which shall be parallel to the straight line. It will also be apparent by drawing a diametral plane through the straight line which will cut the hyperboloids in conjugate hyperbolas and the asymptotic cone in their asymptotes; then, applying well-known properties of the hyperbola, we shall have the theorems (t) at once.

In conclusion, I would observe that the consideration of conjugate hyperboloids seems to be as necessary as that of conjugate hyperbolas. We can obtain a clear geometrical conception of many theorems when enunciated as properties of conjugate hyperboloids, of which we have, I think, but an obscure notion when presented as properties of only *one* of these surfaces. In truth, the few properties of the kind here alluded to, that are usually given in works on Analytical Geometry of Three Dimensions, are enunciated with a tacit reference to the ellipsoid, and the student is afterwards merely informed that the squares of certain quantities are negative in the case of either of the hyperboloids. He is thus furnished with analytical expressions, but with only a very confused idea of their geometrical meaning. The introduction of the conjugate hyperboloid, however, completely dispels this obscurity, and enables us to enunciate such theorems with precision.

ADDENDUM. Since the preceding paper was sketched, it has occurred to me that most of the theorems, (o)....(t), have analogues in reference to the ellipsoid, while some of them are capable of being enunciated with still greater generality. I shall insert these propositions here, but shall omit the investigations (which indeed are not difficult) in order to save space.

DEF. A *conjugate cylinder* circumscribed about an ellipsoid is a cylinder whose ends touch the ellipsoid at the extremities of the diameter parallel to the generators.

(P) The locus of the perimeters of the ends of conjugate cylinders circumscribed about an ellipsoid is a concentric similar ellipsoid whose principal diameters are to those of the given ellipsoid as $\surd 2 : 1$.

(Q) An ellipsoid is two-thirds of each circumscribed conjugate cylinder, and hence all conjugate cylinders circumscribed about the same ellipsoid are equal to one another.

The former part of this proposition is an extension of a property of the sphere.

(O) If elliptic sections of two concentric, similar, and similarly situated surfaces of the second order be made by parallel planes, the elliptic rings bounded by the curves of section will be equal to each other.

(R) Tangent planes to the inner of two concentric, similar, and similarly posited surfaces of the second order cut off equal volumes from the other.

The propositions (O) and (R) hold if the surfaces are either ellipsoids or hyperboloids, and there are analogous properties in respect of two *equal* elliptic paraboloids which have their principal axes in the same straight line and are similarly posited.* The following properties, (T), are true of any of the surfaces of the second order, providing the enunciation be modified as before for the paraboloids.

(T) If there be two concentric, similar, and similarly posited surfaces of the second order and any straight line be drawn cutting the outer surface in U', V', and the other in U'', V''; then $U'U'' = V'V''$, and $U'U''.U''V' = U'V''.V''V'$ is constant for all parallel lines. (When the surfaces are ellipsoids and are related to each other as in (P), $U'U''.U''V' = U'V''.V''V'$ = square of the parallel semidiameter of the inner ellipsoid.)

To be able to perceive that (O), (R), and (T) are extensions of (o), (r), and (t), it must be recollected that a cone is a limiting case of either of the hyperboloids.

Cottenham St., Newcastle-upon-Tyne,
May 4, 1847.

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* That is, providing the equations to the two paraboloids be

$$z + d = \frac{x^2}{p_1} + \frac{y^2}{p_2}, \text{ and } z + d' = \frac{x^2}{p_1} + \frac{y^2}{p_2},$$

the principal axis of each being the axis of z.

NOTES ON HYDRODYNAMICS.

I.—*On the Equation of Continuity.*

By WILLIAM THOMSON.

THE following proof of the Equation of Continuity is simpler than that which is generally given in treatises on Hydrodynamics, and it has also the advantage of shewing in a clearer manner the nature of the property of fluid motion expressed.* Thus, instead of considering a portion of the moving fluid and the varying space which the particles composing it occupy at successive instants, as in the ordinary proof, we imagine a space S fixed in the interior of the fluid, and we consider the fluid which flows into this space, across part of the bounding surface, and that which flows out of it, across the remainder in a given interval of time. The equation of continuity is the analytical expression of the fact that the change in the mean density of the fluid in the space S, during the interval of time considered, is due to the difference between the quantities of fluid which, in that interval, flow into it and out of it, or, if the fluid be of invariable density, that these quantities are equal; and its generality, as applied to all cases of fluid motion, is subject to no exception.

Let the space S be an infinitely small parallelepiped, of which the edges α, β, γ are parallel to the axes of coordinates, and let x, y, z be the coordinates of its centre; so that $x \pm \frac{1}{2}\alpha$, $y \pm \frac{1}{2}\beta$, $z \pm \frac{1}{2}\gamma$ are the coordinates of its angular points. Let ρ be the density of the fluid at (x, y, z), or the mean density through the space S, at the time t. The density at the time $t + dt$ will be $\rho + \frac{d\rho}{dt}dt$; and hence the quantities of fluid contained in the space S, at the times t, and $t + dt$, are respectively $\rho \, . \, \alpha\beta\gamma$ and $\left(\rho + \frac{d\rho}{dt}dt\right)\alpha\beta\gamma$. Hence the quantity of fluid lost (there will of course be an absolute gain if $\frac{d\rho}{dt}$ be positive) in the time dt is

$$-\frac{d\rho}{dt}dt \, . \, \alpha\beta\gamma \ldots\ldots\ldots\ldots\ldots (a).$$

* Poisson admits that the proof which he gives is inapplicable to certain conceivable circumstances of fluid motion; but he erroneously concludes that in such cases the equation "of continuity" does not hold. (See Poisson's *Traité de Mécanique*, No. 651.) The proof in the text has been frequently given in lectures at Cambridge, and elsewhere, and it is likely to occur to any one reading Fourier's Theory of Heat; but I am not aware that it has been hitherto published in any work except Duhamel's *Cours de Mécanique* (Deuxième Partie; Paris 1847).

Now let u, v, w be the three components of the velocity of the fluid (or of a fluid particle*) at P. These quantities will be functions of x, y, z, (involving also t, except in the case of "steady motion,") and will in general vary gradually from point to point of the fluid; although the analysis which follows is not restricted by this consideration, but holds even in cases where in certain places of the fluid there are abrupt transitions in the velocity, as may be seen by considering them as limiting cases of motions in which there are very sudden continuous transitions of velocity. If ω be a small plane area, perpendicular to the axis of x, and having its centre of gravity at P, the volume of fluid which flows across it in the time dt will be equal to $u\,.\,\omega\,.\,dt$, and the mass or quantity will be $\rho\,.\,u\,.\,\omega\,.\,dt$. If we substitute $\beta\gamma$ for ω, the quantity which flows across the either of the sides $\beta\gamma$ of the parallelepiped S, will differ from this only on account of the variation in the value of ρu; and therefore the quantities which flow across the two sides $\beta\gamma$ are respectively

$$\left\{\rho u - \tfrac{1}{2}\,a\,\frac{d(\rho u)}{dx}\right\}.\,\beta\gamma\,.\,dt,$$

and

$$\left\{\rho u + \tfrac{1}{2}\,a\,\frac{d(\rho u)}{dx}\right\}.\,\beta\gamma\,.\,dt.$$

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* This explanatory clause must be omitted, and a modified definition of fluid velocity must be given, if it be required to include the case, imagined by Poisson, of the motion of two fluids of different densities *through* one another, in which, as he conceives, the "*molécules*" of the lighter fluid will move upwards, between the "*molécules*" of the heavier fluid, which descend. Thus we should define u as the mean velocity of the "*molécules*" parallel to the axis of x, across any very small plane of which the centre of gravity is at P, and we should thus obtain the same equation as that found in the text, which is applicable to this as to every possible case of fluid motion. It is however very doubtful whether this kind of motion can actually exist in nature. In the case, considered by Poisson (Art. 651), of water contained in a vertical cylinder, open above, and heated at its bottom which is supposed to be horizontal, it is certainly true that the regular upward motion of the whole fluid due to the expansion of the lower strata is practically impossible, because unstable; but as far as experience indicates, (by the *streaks* we can see on looking into the vessel, on account of the varying refracting power of the heterogeneous liquid,) we find that the effect of the instability is to disturb the surfaces of equal density from being horizontal planes, and thus to allow finite portions of the lighter fluid to ascend, their places being filled by the heavier fluid descending. The definition, in the text, of the components, u, v, w, is directly applicable to this kind of motion; and the analysis and resulting equation, when interpreted according to the principles of the differential calculus as applied to discontinuous functions, will not be subject to exception even in cases when the ascending and descending portions slide upon one another with finite velocities; cases which might actually occur were there no "friction of fluids in motion."

Hence $\alpha\,\dfrac{d(\rho u)}{dx}\,.\,\beta\gamma\,.\,dt$, or $\dfrac{d(\rho u)}{dx}\,.\,\alpha\beta\gamma\,.\,dt$ is the excess of the quantity of fluid which leaves the parallelepiped across one of the faces $\beta\gamma$ above that which enters it across the other. By considering in addition the effect of the motion across the other faces of the parallelepiped, we find for the total quantity of fluid lost from the space S, in the time dt,

$$\left\{\frac{d(\rho u)}{dx}+\frac{d(\rho v)}{dy}+\frac{d(\rho w)}{dz}\right\}.\,\alpha\beta\gamma\,.\,dt\,\ldots\ldots(b).$$

Equating this to the expression (a), previously found, we have

$$\left\{\frac{d(\rho u)}{dx}+\frac{d(\rho v)}{dy}+\frac{d(\rho w)}{dz}\right\}.\,\alpha\beta\gamma\,.\,dt=-\frac{d\rho}{dt}\,.\,dt\,.\,\alpha\beta\gamma\,;$$

and we deduce

$$\frac{d(\rho u)}{dx}+\frac{d(\rho v)}{dy}+\frac{d(\rho w)}{dz}+\frac{d\rho}{dt}=0\,\ldots\ldots\ldots(1),$$

which is the required equation.

If, instead of taking the infinitely small parallelepiped $\alpha\beta\gamma$, and the infinitely small interval of time dt, we consider a finite space S bounded by a fixed surface, and a finite interval of time, from t_1 to t_2, the equation of continuity should, it is clear from the demonstration given above, express this fact; that the mass of fluid in S at the time t_2 is equal to the mass at the time t_1, wanting the total mass which has been taken away by the flux across the surface. This is verified directly by the following analytical process.*

Let ρ_1, ρ_2 be the densities of the fluid at (xyz) at the times t_1, t_2, and let M_1, M_2 be the total masses contained at those times in the space S. We shall have

$$M_1=\iiint\rho_1\,dx\,dy\,dz,\quad M_2=\iiint\rho_2\,dx\,dy\,dz,$$

and therefore $M_2=M_1+\iiint(\rho_2-\rho_1)\,dx\,dy\,dz$

$$=M_1+\int_{t_1}^{t_2}dt\iiint\frac{d\rho}{dt}\,dx\,dy\,dz\,\ldots\ldots\ldots(2).$$

But, by the equation of continuity, we find

$$-\iiint\frac{d\rho}{dt}\,dx\,dy\,dz=\iiint\left\{\frac{d(\rho u)}{dx}+\frac{d(\rho v)}{dy}+\frac{d(\rho w)}{dz}\right\}dx\,dy\,dz,$$

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* Compare *Camb. Math. Jour.* vol. III. p. 203; also Poisson, *Théorie de la Chaleur*, p. 177.

and hence, by separating the second member into three terms, and performing the integrations, in the first with respect to x, in the second with respect to y, and in the third with respect to z, and assigning the limits so as to include the whole space S, we have

$$-\iiint \frac{d\rho}{dt}\,dx\,dy\,dz = \iint \rho u.\,dy\,dz + \iint \rho v.\,dz\,dx + \iint \rho w.\,dx\,dy \ldots (3),$$

where the values of xyz in each term of the second member belong to the surface of S. Now let ds be an element of the surface at xyz, and let l, m, n be the direction cosines of a normal: we may take ds such that

$$ds.l = dy\,dz, \quad ds.m = dz\,dx, \quad ds.n = dx\,dy\,;$$

and modifying accordingly the second member of (3), we have

$$-\iiint \frac{d\rho}{dt}\,dx\,dy\,dz = \iint \rho\,(lu + mv + nw)\,ds \ldots (4).$$

Hence (2) becomes

$$M_2 = M_1 - \int_{t_1}^{t_2} dt \iint \rho\,(lu + mv + nw)\,ds,$$

or

$$M_2 = M_1 - \iint ds. \int_{t_1}^{t_2} \rho\,(lu + mv + nw)\,dt \ldots\ldots (5).$$

Now $lu + mv + nw$ is the component of the velocity of the fluid in the direction of the normal at xyx, and therefore $ds.\int_{t_1}^{t_2} \rho\,(lu + mv + nw)\,dt$ is the quantity which flows out of the space S, across the element ds, of the surface, in the interval considered. Hence the total quantity, lost from S in the time $t_2 - t_1$, is equal to the integral in the second member of equation (5), and this equation is therefore the expression of the required result.

If the mass we are considering be a liquid (that is to say, an incompressible fluid), even although it be heterogeneous, the equation of continuity assumes a simpler form. For the density at a point xyz, moving with the fluid, will be invariable, and therefore the differential of ρ, considered as a function of x, y, z, t, will vanish provided we take $dx = udt$, $dy = vdt$, $dz = wdt$. Hence

$$\frac{d\rho}{dt}\,dt + \frac{d\rho}{dx}.udt + \frac{d\rho}{dy}.vdt + \frac{d\rho}{dz}.wdt = 0.$$

Dividing by dt, then subtracting the first member from that of the equation of continuity, and dividing the result by ρ, we find

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \; \ldots\ldots\ldots\ldots \; (6).$$

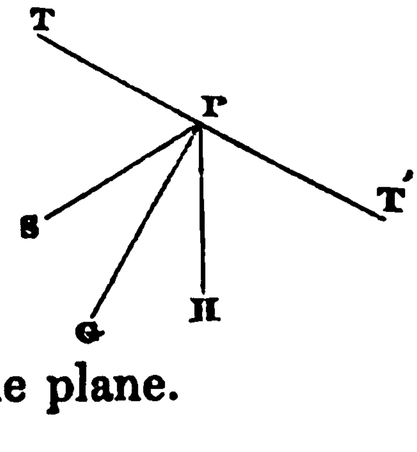
This equation has the same form as in the case when the liquid is homogeneous, as might easily have been proved directly, by considering merely the volume of fluid which flows through the space S, and not its mass, which, when the fluid is incompressible, will enable us to arrive at the equation of continuity.

St. Peter's College, Sept. 29, 1847.

MATHEMATICAL NOTE.

On the Maximum or Minimum Property of Incident and Reflected Rays.

The following is a very simple analytical proof of the proposition, that when a ray of light is reflected at any surface, the length of the path of the ray, measured from a given point in the incident to a given point in the reflected ray, is less than it would be according to any law of reflexion other than the actual law.



Let P be the point of incidence, SP the incident, PH the reflected ray, PG the normal to the surface at P.

Then we have to prove, that if $SP + PH$ is a minimum,

then (1) SP, PH, PG are in the same plane.

(2) $SPG = HPG$.

Let $x\,y\,z$ be the coordinates of P,

$\alpha\,\beta\,\gamma$ .... S,

$\alpha'\beta'\gamma'$ .... H.

$SP = r$, $PH = r'$. Then the condition that $r + r'$ shall be a minimum gives us

$$dr + dr' = 0,$$

or $$\frac{dr}{dx}\,dx + \frac{dr}{dy}\,dy + \frac{dr}{dz}\,dz + \frac{dr'}{dx}\,dx + \frac{dr'}{dy}\,dy + \frac{dr'}{dz}\,dz = 0 \ldots (\mathrm{A}).$$

Now it will be easily seen that the equations of SP, PH, PG, are respectively

$$\frac{x_1 - x}{\frac{dr}{dx}} = \frac{y_1 - y}{\frac{dr}{dy}} = \frac{z_1 - z}{\frac{dr}{dz}},$$

$$\frac{x_1 - x}{\frac{dr'}{dx}} = \frac{y_1 - y}{\frac{dr'}{dy}} = \frac{z_1 - z}{\frac{dr'}{dz}},$$

$$\frac{x_1 - x}{\frac{df}{dx}} = \frac{y_1 - y}{\frac{df}{dy}} = \frac{z_1 - z}{\frac{df}{dz}},$$

f being such a function that $f(x, y, z) = 0$ is the equation of the given surface. In order that these may lie in the same plane, we must have

$$\frac{df}{dx}\left(\frac{dr}{dy}\frac{dr'}{dz} - \frac{dr}{dz}\frac{dr'}{dy}\right) + \frac{df}{dy}\left(\frac{dr}{dz}\frac{dr'}{dx} - \frac{dr}{dx}\frac{dr'}{dz}\right) + \frac{df}{dz}\left(\frac{dr}{dx}\frac{dr'}{dy} - \frac{dr}{dy}\frac{dr'}{dx}\right) = 0 \ldots \text{(B)}.$$

But $$\frac{df}{dx}dx + \frac{df}{dy}dy + \frac{df}{dz}dz = 0,$$

and this equation and (A) are the only relations between dx, dy, dz; hence we may take

$$\frac{df}{dx} = \lambda\left(\frac{dr}{dx} + \frac{dr'}{dx}\right),$$

$$\frac{df}{dy} = \lambda\left(\frac{dr}{dy} + \frac{dr'}{dy}\right),$$

$$\frac{df}{dz} = \lambda\left(\frac{dr}{dz} + \frac{dr'}{dz}\right).$$

If we substitute these values, (B) assumes the form

$$\left(\frac{dr}{dx} + \frac{dr'}{dx}\right)\left(\frac{dr}{dy}\frac{dr'}{dz} - \frac{dr}{dz}\frac{dr'}{dy}\right) + \left(\frac{dr}{dy} + \frac{dr'}{dy}\right)\left(\frac{dr}{dz}\frac{dr'}{dx} - \frac{dr}{dx}\frac{dr'}{dz}\right) + \left(\frac{dr}{dz} + \frac{dr'}{dz}\right)\left(\frac{dr}{dx}\frac{dr'}{dy} - \frac{dr}{dy}\frac{dr'}{dx}\right) = 0,$$

which being an identical equation, the first part of the proposition is true.

The second follows very simply, for let TPT' be the intersection of the tangent plane with the plane SPH; then if ds be an element of the line TPT',

$$\cos SPT = \frac{dr}{dx}\frac{dx}{ds} + \frac{dr}{dy}\frac{dy}{ds} + \frac{dr}{dz}\frac{dz}{ds},$$

and $$\cos HPT = \frac{dr'}{dx}\frac{dx}{ds} + \frac{dr'}{dy}\frac{dy}{ds} + \frac{dr'}{dz}\frac{dz}{ds};$$

therefore, by the fundamental equation (A),

$$\cos SPT + \cos HPT = 0,$$

or $$SPT = 180° - HPT = HPT',$$

and hence $$SPG = HPG,$$

which is the second part of the proposition.

H. G.

Cambridge, Sept. 29, 1847.

[The following geometrical proof, although not new, may be added, in connection with the preceding.

With S and H as foci, and SH as axis of revolution, describe a prolate spheroid, touching the reflecting surface in P. Then SPH is the course of the incident and reflected ray, since the plane SPH, passing through the axis of the spheroid, is perpendicular to the tangent plane at P, and SP, PH, by the known property of the ellipse, make equal angles with the normal PG. Now the value of $SQ + QH$, for any point Q, without the spheroid, is, as follows from another well-known property of the ellipse, greater than $SP + PH$. Hence if the spheroid is touched externally by the reflecting surface, the actual course of the incident and reflected ray is less than if the point of incidence on the surface were in any other position Q, in the neighbourhood of P. We see also that, in general, the point of incidence, P, on the surface is determined by the maximum or minimum condition; although in some cases $SP + PH$ may be actually a maximum, and in others neutral.]

END OF VOL. II.

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$$1.2\ldots n = n^n - n(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n + \ldots etc.$$

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